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Autor: Shephard, G. C.

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SOME PROBLEMS ON FINITE REFLECTION GROUPS

by

G. C. SHEPHARD, Birmingham

§ 1. INTRODUCTION.

Let V^r be an r dimensional vector space over some field k of zero characteristic. By a *hyperplane* we mean a linear subspace of V^r of $r - 1$ dimensions. A linear transformation on V^r that is not the identity is called a *reflection* if it leaves some hyperplane pointwise invariant and is of finite order. If $k = R$, the real numbers, then every reflection must be of order 2. Let W be a finite group of linear transformations on V^r such that the elements of W which are reflections generate W. Then W is said to be a *finite* r *dimensional group generated by reflections* or, more briefly, a *reflection group.*

The purely geometrical properties of reflection groups over R have been discussed at length by H. S. M. COXETER and other authors (see the bibliography of [5]) and some of these have been extended by the author [8] to the case $k = C$, the complex numbers. The irreductible reflection groups over C have been enumerated, the complete list is given in [9; p. 301].

In this note we are primarily concerned with the algebraic properties of reflection groups. The first theorem due to Chevalley (see (i) of §3) states that every polynomial that is invariant under the transformations of a reflection group can be expressed as a polynomial in a set of r *basic invariant forms* $I_1, I_2, \ldots, I_r$. Writing $m_i + 1$ for the degree of I_i, the r integers m_i are called the *exponents* of the group W. The remaining theorems of § 3 are, in effect, properties of these integers. Several of these were first noticed by COXETER [6] for $k = R$ and further ones (together with the extension of COXETER's results to the complex numbers) are due to J. A. TODD and the author [9]. More recently the work of BOTT [2] has given a new interpretation ((iv) of § 3) to the exponents of a *crystallographic* group of reflections over R, connecting them with the diagram of the corresponding Lie Group.

In stating the theorems in § 3, we give explicitly the restrictions that must be placed on the ground field k and also on the group W. For each theorem it is briefly indicated how the result may be proved. Sometimes a proof in general terms is known, but in the majority of cases it has been necessary to verify the properties one by one for all the irreducible groups over C and then show that (with the exception of (vi)) they extend to the reducible groups. These two distinct methods will be referred to as *proving* and *verifying* respectively. Perhaps the most remarkable fact is that the result (iv) of § 3 which appears to concern itself entirely with discrete infinite groups generated by reflections has been proved only by topological methods (spectral sequences and Morse theory). A direct proof, avoiding the topology, would be interesting. Further outstanding problems are the discovery of proofs for those properties that have so far only been verified, and the extension of these theorems to more general fields k, especially to the case where k is of finite characteristic.

§ 2. The Connection between Lie Groups and Reflection Groups.

Let G be an n dimensional compact semi-simple Lie Group. A maximal connected abelian subgroup of G forms a submanifold of G which is a torus of dimension r (the *rank* of G) [10]. This is called a *maximal torus* T of G. The inner automorphisms of G by elements of N_T, the normaliser of T, induce a finite group of automorphisms of T. These in turn induce linear transformations of the tangent space V^r to T at the identity e. It can be shown that this group of linear transformations forms a reflection group over R called the *Weyl group* W of G. This group has the further property that it is *crystallographic*, i.e. by suitable choice of coordinates it is represented by a set of matrices whose coefficients are integers, or, alternatively, if the coordinates are chosen so that the matrices are orthogonal (so that W is then a group of congruent transformations acting on a Euclidean space R^r) then the angle between any two hyperplanes of reflection of W is an integral multiple of $\pi/4$ or $\pi/6$.

The converse is also true, namely that any crystallographic reflection group over R corresponds to some compact semi-simple Lie Group.

It can be shown that T may be covered by a Euclidean space R^r in such a way that the *singular elements* of T (i.e. those whose normalisers are of dimension strictly greater than r) map into hyperplanes of R^r, and further, if the identity of G maps into the origin O of R^r, then those planes passing through O are precisely the hyperplanes of reflection of the Weyl group W. The whole set of hyperplanes form a configuration known as the *diagram* of the Lie Group G and it has the property that reflection in any one of the planes leaves the diagram, as a whole invariant.

Now let W be *any* reflection group over R expressed in orthogonal form, then W may be considered as operating on some sphere S^{r-1} whose centre is at O. The hyperplanes of reflection divide the surface of the sphere into spherical polytopes and it has been shown [5; p. 190] that each of these is necessarily a simplex or a direct product of simplexes. Further, the r hyperplanes that cut the sphere in the faces of one of these polytopes form a *fundamental set* in that the corresponding reflections generate the group. Furthermore, the volume bounded by these hyperplanes forms a fundamental region for W. A property of this fundamental set is given in (vi) of § 3.

Considering again the diagram of the Lie Group G, pick out a fundamental set of hyperplanes through O, defining a fundamental region of the Weyl group. Then the part of the diagram of G that lies within this fundamental region is called a *Weyl chamber.* The Weyl chamber of the group G_2 (the group of automorphisms of the Cayley matrix algebra) is illustrated in (iv) of § 3, where, for the present, the numerals are to be ignored.

§ 3. Properties of the Exponents.

(i) The ring of polynomial invariants of an r dimensional reflection group W over k is the ring $k[I_1, I_2, \ldots, I_r]$ where I_i is a polynomial invariant of degree $(m_i + 1)$. The I_i are uniquely determined by this property and are called the *basic invariants* of the group W. The m_i are called the *exponents* of W.

This theorem was proved by CHEVALLEY for k of characteristic zero [4] and a partial converse (for $k = C$) by TODD [9; p. 282].

From this result we can draw several conclusions. Firstly it implies a formal identity in power series in t:

$$\prod_{i=1}^{r} \frac{1}{(1 - t^{m_i+1})} = \sum_{j=0}^{\infty} a_j t^j = \frac{1}{w} \sum_{S \varepsilon W} \frac{1}{|I - tS|}$$

where w is the order of W, and a_j is the number of linearly independent polynomial invariants of W of degree j (which, by the above result, is the number of monomials in the I_i of total degree j, and hence is the coefficient of t^j in the expansion of the product on the left). The identity on the right is the classical result of MOLIEN [3; p. 300]. Considering only the first and last terms of this identity, after a small amount of manipulation we deduce

(a) $\prod_{i=1}^{r} (1 + m_i) = w$

(b) $\sum m_i = b_1 =$ number of reflections in W.

(ii) More generally there is an identity

$$\prod_{i=1}^{r} (1 + m_i t) = \sum_{j=1}^{r} b_j t^j$$

where b_j is the number of transformations in W that leave pointwise invariant a linear subspace of V^r of exactly $(r - j)$ dimensions. Putting $t = 1$ we obtain *(a)* above, and *(b)* merely states the equality of coefficients of t on both sides. No general proof is known, but this result has been verified for all reflection groups over C.

(iii) Writing $H^*(X, R)$ for the cohomology ring of a space X with coefficients in R, S^n for the n-sphere, and $\Lambda(x_1, x_2, \ldots, x_r)$ for the graded exterior algebra on r generators, then for any compact semi-simple Lie group G, we have

$$H^*(G, R) = H^*(S^{2m_1+1} \times S^{2m_2+1} \times \ldots \times S^{2m_r+1}, R),$$

$$= \Lambda(x_1, x_2, \ldots, x_r), \quad \partial^0 x_i = 2m_i + 1.$$

This is the celebrated Hopf theorem [7] except for the number and dimensions of the spheres (or of the generators of Λ). The relation between the dimensions and the exponents of W is due to Chevalley [4]. An independent proof has been given by A. Borel [1].

(iv) Consider the Weyl chamber of a connected semi-simple Lie group G, whose vertex is O. Then each simplex Δ_r of the chamber is to be labelled with an integer equal to the number of intersections with the hyperplanes of a line joining O to an interior point of Δ_r. It is trivially verified that such an integer is uniquely defined, i.e. is independent of the interior point chosen. Then we have

$$\prod_{i=0}^{r} \frac{1}{(1 - t^{m_i})} = \sum_{i=0}^{\infty} c_i t^i$$

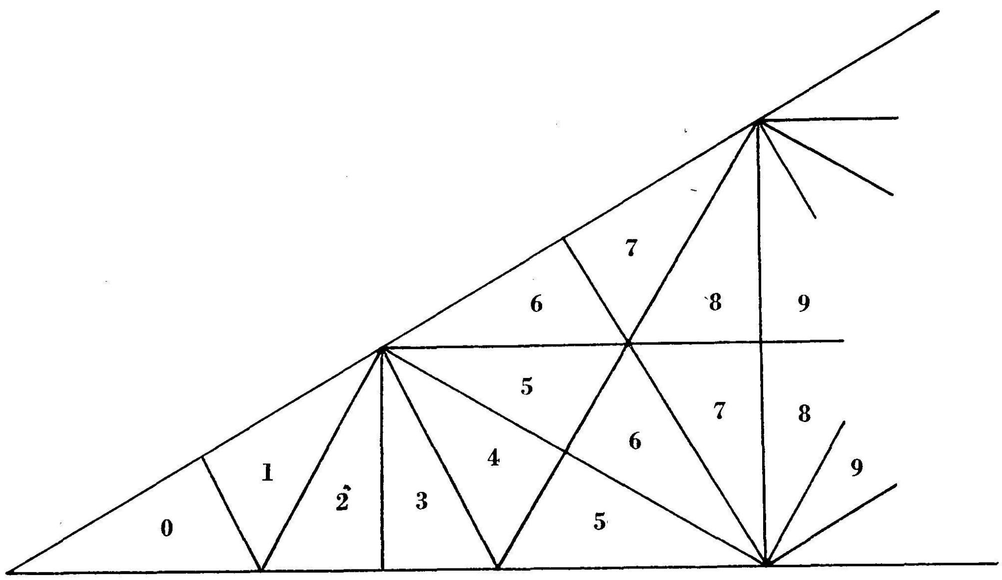
where c_i is the number of simplexes in the Weyl chamber labelled with the integer i.

By way of example the Weyl chamber of G_2 is illustrated below. Here,

$$c_0 = c_1 = c_2 = c_3 = c_4 = 1 ,$$

$$c_5 = c_6 = c_7 = c_8 = c_9 = 2 , \text{ etc.}$$

The exponents are $m_1 = 1$ and $m_2 = 5$.



The statement of this result is only meaningful if $k = R$ and W is crystallographic, and then it can be proved as follows:

From (iii) by the theory of spectral sequences

$$H^* (\Omega_G , R) = R [u_1 , u_2 , \ldots, u_r] , \qquad \partial^0 u_i = 2 m_i$$

where the right side is the graded ring of polynomials over R and Ω_G is the space of loops on G. Hence if we write $P(\Omega_G, t)$ (the Poincaré polynomial of Ω_G in t) for $\sum_{i=0}^{\infty} t^i \dim H^i (\Omega_G , R)$ we have

$$P (\Omega_G , t) = \prod_{i=1}^{r} \frac{1}{(1 - t^{2mi})}$$

Bott, by the use of Morse theory [2] proved that

$$\sum_{i=0}^{\infty} c_i t^{2i} = P (\Omega_G, t)$$

from which the given result follows immediately.

(v) The Jacobian of the basic invariants I_i,

$$J = \frac{\partial (I_1 , I_2 , \ldots, I_r)}{\partial (x_1 , x_2 , \ldots, x_r)}$$

factorises into Σm_i linear forms, which, when equated to zero, give the hyperplanes of reflection of W, and each hyperplane is repeated $p - 1$ times where p is the order of the corresponding reflection.

Where all the reflections are of order 2, a very simple argument proves this result in a more rigorous manner than that of Racah [6; p. 775]. For *(b)* of (i) implies that the degree of J is equal to the number of reflections, and the fact that J changes sign when operated on by a reflection in W implies that the equation of each hyperplane is a factor of J. This proves the result. More generally it can be proved over any field of zero characteristic for reflections of any order.

An interesting conjecture extends this result. In (ii) we defined b_j as the number of transformations in the group that leave a linear subspace of $n - j$ dimensions invariant. The set of all these linear subspaces forms a reducible algebraic variety of dimension $n - j$ and of degree b_j, and it is conjectured that this is given by equating to zero all the $(n - j + 1)$-rowed minors of the functional matrix $\partial I_i / \partial x_j$ This conjecture has neither been proved or verified.

(vi) This final property is the only one that holds for *irreducible* groups only. Suppose that r reflections in W serve to generate W (this is always the case for k = R, but not for k = C). Then it is possible to pick this set of generating reflections so that their product has characteristic roots

$$\exp\left(\frac{2 \pi i m_j}{h}\right), j = 1, 2, \ldots, r; h = \max(m_j) + 1$$

When k = R it suffices to choose the reflections as those of a fundamental set and then take their product in any order [6; p. 765]. In this case also h has geometric significance as the number of sides of the Petrie polygon [5; p. 223]. For k = C no general rule for the selection of the correct set of reflections has been given.

This result has been verified for k = C, and general proofs are known for k = R, r = 2, 3 [6; p. 772].

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