§ 1. Introduction.

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SOME PROBLEMS ON FINITE REFLECTION GROUPS

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§ 1. INTRODUCTION.

Let V^r be an r dimensional vector space over some field k of zero characteristic. By a *hyperplane* we mean a linear subspace of V^r of $r - 1$ dimensions. A linear transformation on V^r that is not the identity is called a *reflection* if it leaves some hyperplane pointwise invariant and is of finite order. If $k = R$, the real numbers, then every reflection must be of order 2. Let W be a finite group of linear transformations on V^r such that the elements of W which are reflections generate W. Then W is said to be a *finite* r *dimensional group generated by reflections* or, more briefly, a *reflection group.*

The purely geometrical properties of reflection groups over R have been discussed at length by H. S. M. COXETER and other authors (see the bibliography of [5]) and some of these have been extended by the author [8] to the case $k = C$, the complex numbers. The irreductible reflection groups over C have been enumerated, the complete list is given in [9; p. 301].

In this note we are primarily concerned with the algebraic properties of reflection groups. The first theorem due to Chevalley (see (i) of §3) states that every polynomial that is invariant under the transformations of a reflection group can be expressed as a polynomial in a set of r *basic invariant forms* $I_1, I_2, \ldots, I_r$. Writing $m_i + 1$ for the degree of I_i, the r integers m_i are called the *exponents* of the group W. The remaining theorems of § 3 are, in effect, properties of these integers. Several of these were first noticed by COXETER [6] for $k = R$ and further ones (together with the extension of COXETER's results to the complex numbers) are due to J. A. TODD and the author [9]. More recently the work of BOTT [2] has given a new interpretation ((iv) of § 3) to the exponents of a *crystallographic* group of reflections over R, connecting them with the diagram of the corresponding Lie Group.

In stating the theorems in § 3, we give explicitly the restrictions that must be placed on the ground field k and also on the group W. For each theorem it is briefly indicated how the result may be proved. Sometimes a proof in general terms is known, but in the majority of cases it has been necessary to verify the properties one by one for all the irreducible groups over C and then show that (with the exception of (vi)) they extend to the reducible groups. These two distinct methods will be referred to as *proving* and *verifying* respectively. Perhaps the most remarkable fact is that the result (iv) of § 3 which appears to concern itself entirely with discrete infinite groups generated by reflections has been proved only by topological methods (spectral sequences and Morse theory). A direct proof, avoiding the topology, would be interesting. Further outstanding problems are the discovery of proofs for those properties that have so far only been verified, and the extension of these theorems to more general fields k, especially to the case where k is of finite characteristic.

§ 2. The Connection between Lie Groups and Reflection Groups.

Let G be an n dimensional compact semi-simple Lie Group. A maximal connected abelian subgroup of G forms a submanifold of G which is a torus of dimension r (the *rank* of G) [10]. This is called a *maximal torus* T of G. The inner automorphisms of G by elements of N_T, the normaliser of T, induce a finite group of automorphisms of T. These in turn induce linear transformations of the tangent space V^r to T at the identity e. It can be shown that this group of linear transformations forms a reflection group over R called the *Weyl group* W of G. This group has the further property that it is *crystallographic*, i.e. by suitable choice of coordinates it is represented by a set of matrices whose coefficients are integers, or, alternatively, if the coordinates are chosen so that the matrices are orthogonal (so that W is then a group of congruent transformations acting on a Euclidean space R^r) then the angle between any two hyperplanes of reflection of W is an integral multiple of $\pi/4$ or $\pi/6$.