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4. THE NUMBER OF ZEROS.

THEOREM 2. For $n \geq 1$ let $z_{k,n}$ be the probability that exactly k among the n partial sums S_1, \dots, S_n vanish. For $n = 0$ put

$$(4.1) \quad z_{0,0} = 1, \quad z_{1,0} = z_{2,0} = \dots = 0.$$

Then

$$(4.2) \quad z_{k,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n}.$$

Proof. By definition

$$(4.3) \quad z_{0,2n} = p_{2n} = u_{2n}, \quad (n \geq 0).$$

To evaluate $z_{1,2n}$ denote by B_r the event that among the partial sums S_1, \dots, S_{2n} exactly one vanishes and its index equals $2r$. Then for $r < n$

$$\begin{aligned} B_r &= \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2r+1} \neq 0, \dots, S_{2n} \neq 0 \} \\ &\equiv \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0, \dots, S_{2n} - S_{2r} \neq 0 \}. \end{aligned}$$

Since the two events on the right are stochastically independent and the B_r are mutually exclusive we conclude that

$$(4.5) \quad z_{1,2n} = \sum_{r=1}^n P \{ B_r \} = \sum_{r=1}^n f_{2r} z_{0,2n-2r}.$$

Now by Theorem 1 the last event on the right in (4.4) has the same probability as the event $\{ S_{2n} - S_{2r} = 0 \}$ and hence we have for $r \leq n$

$$(4.6) \quad P \{ B_r \} = P \{ S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0 \}.$$

The events appearing on the right side are mutually exclusive and their union is the event $\{ S_{2n} = 0 \}$; hence

$$(4.7) \quad \sum_{r=1}^n P \{ B_r \} = P \{ S_{2n} = 0 \} = u_{2n}.$$

Comparing (4.5) and (4.7) we see that

$$(4.8) \quad z_{1,2n} = u_{2n} = z_{0,2n} \quad \text{for } n \geq 1.$$

In like manner we can calculate $z_{2,2n}, z_{3,2n}, \dots$ from the recursion formula

$$(4.9) \quad z_{k,2n} = \sum_{r=1}^{n-1} f_{2r} z_{k-1,2n-2r}, \quad k \geq 2, \quad n \geq 1.$$

which is proved exactly as (4.5). For $k \geq 2$ the right side differs from the right side in (4.5) only in that the term $r = n$ is absent, and therefore

$$(4.10) \quad z_{k,2n} = z_{1,2n} - f_{2n} = 2z_{1,2n} - z_{0,2n-2}, \quad n \geq 1.$$

From the last two relations we see directly by induction that for $k \geq 2$ and $n \geq 1$ we have the recursion formula

$$(4.11) \quad z_{k,2n} = 2z_{k-1,2n} - z_{k-2,2n-2}$$

If we write $z_{k,2n} = 2^{k-2n} a_{k,2n}$ then (4.11) reduces to

$$(4.12) \quad a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

COROLLARY. For each $n \geq 1$ we have

$$(4.13) \quad z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \dots > z_{n,2n}$$

Thus, independently of the number n of steps, the *most probable number of zeros* is 0, and the smaller the number, the more probable it is.

5. THE NUMBER OF CHANGES OF SIGN.

We say that in the sequence S_1, \dots, S_{2n} a *change of sign occurs at the place j* if S_{j-1} and S_{j+1} are of opposite signs. This requires that $S_j = 0$, and so j must be even. Given the first $2n$ terms