4. The Functions $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ as Double Sums.

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which is equivalent to a known recurrence for the BERNOULLI numbers [8].

4. THE FUNCTIONS $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ AS DOUBLE SUMS.

The results which are stated as (4), (5), (6) follow readily from (1) and (2) which are known to be equivalent (see [1], [2]). It is to be observed first that a comparison of (4) and (5) with (1) taking into account (27) gives the relations:

$$(30) \qquad \Psi_{2k-1}(t) = h_{2k-1}(t/2) - h_{2k-1}(t) = \mathrm{V}_k\left(\alpha_{2k-1}(t/2) - \alpha_{2k-1}(t)\right),$$

$$(31) \qquad X_{2k-1}(t) = 2^{2k} h_{2k-1}(2t) - h_{2k-1}(t) = \mathrm{V}_k\left(2^{2k}\alpha_{2k-1}(2t) - \alpha_{2k-1}(t)\right).$$

From (4) and (6) we also have,

$$(32) \qquad \Phi_{2k-1}(t) = 2^{2k}\Psi_{2k-1}(2t) - \Psi_{2k-1}(t).$$

By (30), we may write

$$(33) \qquad \Phi_{2k-1}(t) = -\mathrm{V}_k\left(\alpha_{2k-1}(t/2) - (2^{2k}+1)\,\alpha_{2k-1}(t) + 2^{2k}\alpha_{2k-1}(2t)\right).$$

Thus, our functions (4), (5), (6) are expressed in terms of $\alpha_{2k-1}(u)$. These relations in conjunction with (1) and (2) identify them with $(4)_1$, $(5)_1$, and $(6)_1$ respectively.

It is of interest to note that (31) with $k = 2$ permits, with the aid of a result of VAN DER POL [1], the deduction of Jacobi's famous theorem on the number of representations $r_8(n)$ of the integer n as the sum of eight squares. Thus,

$$(34) \qquad 240\, X_3(t) = 16\,\alpha_3(2t) - \alpha_3(t) = 15\,\theta_0^8(0, q)$$

where $q = \exp(-t)$. Hence,

$$\theta_0^8(0, q) = 16\, X_3(t) = 1 + 16\sum_{n=1}^{\infty} q^n \zeta_3(n),$$

and

$$\theta_3^8(0, q) = 1 + 16\sum_{n=1}^{\infty} (-1)^n q^n \zeta_3(n).$$

This result implies that

$$r_8(n) = 16\,(-1)^n\,\zeta_3(n) = 16\,(-1)^{n-1}\left(\sigma_3^0(n) - \sigma_3^e(n)\right), \tag{35}$$

where $\sigma_3^0(n)$ denotes the sum of the third powers of the odd divisors of n, and $\sigma_3^e(n)$ denotes the sum of the third powers of the even divisors of n. This is the desired result. [8]

5. Modular Transforms.

It has been shown in [2] that for $k > 1$, the function $\alpha_{2k-1}(t)$ satisfies the modular transformation

$$t^k\,\alpha_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k}\,\alpha_{2k-1}(2\pi/t)\,. \tag{36}$$

For $k = 1$, the conditional convergence of the double series in (1) creates difficulties [9], which however, have been resolved by Hurwitz [3], who gives a result equivalent, in our notation, to the formula

$$t\,\alpha_1(2\pi t) = -\frac{1}{t}\,a_1(2\pi/t) + \frac{6}{\pi}\,. \tag{37}$$

We find that this result may be proved very easily by using (36) in conjunction with the relation

$$\alpha_5(t) = \alpha_3'(t) + \alpha_1(t)\,\alpha_3(t)\,, \tag{38}$$

which is the case $n = 2$ in (26).

With the aid of equations (30), (31) and (33), the transforms (36) and (37) yield those for our functions $(4)_1$, $(5)_1$ and $(6)_1$. It is found that under the modular transformation in question, the first two functions are reciprocal in the sense that,

$$t^k\,\Psi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k}\,\chi_{2k-1}(2\pi/t)\,,\quad k \geq 1\,. \tag{39}$$

The remaining function (6), transforms in a manner analogous to $\alpha_{2k-1}(t)$, namely

$$t^k\,\Phi_{2k-1}(2\pi t) = \frac{(-1)^k}{t^k}\,\Phi_{2k-1}(2\pi/t)\,,\quad k > 1\,, \tag{40}$$