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# A SPECIAL CASE OF KUMMER'S CONGRUENCE

(mod  $2^e$ )

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(Reçu le 30 janvier 1959.)

Put

$$\frac{x}{\sinh x} = \sum_1^{\infty} D_{2n} \frac{x^{2n}}{(2n)!}, \quad D_{2n+1} = 0, \quad (1)$$

where [3, p. 27-28]

$$(D + 1)^n - (D - 1)^n = \begin{cases} 2 & (n = 1) \\ 0 & (n < 1); \end{cases}$$

also

$$D^n = (2B + 1)^n = (2 - 2^n) B_n \quad (2)$$

and  $B_n$  is defined by

$$\frac{x}{e^x - 1} = \sum_0^{\infty} B_n \frac{x^n}{n!}.$$

The first few values of  $D_{2n}$  are

$$D_0 = 1, \quad D_2 = -\frac{1}{3}, \quad D_4 = \frac{7}{15}, \quad D_6 = \frac{31}{21}, \quad D_8 = \frac{127}{15}.$$

If we define

$$\Delta_n = (D^2 - 1)^n = \sum_{r=0}^n (-1)^{n-r} \binom{n}{r} D_{2r} \quad (3)$$

we find that

$$\Delta_0 = 1, \quad \Delta_1 = \frac{2^2}{3}, \quad \Delta_2 = \frac{2^5}{15}, \quad \Delta_3 = \frac{2^9}{105}, \quad \Delta_4 = \frac{2^{11}}{105}.$$

(Compare [1, p. 821].)

By the second part of (2), both numerator and denominator of  $D_{2n}$  (in reduced form) are odd. Moreover by the Staudt-Clausen theorem for the Bernoulli numbers  $B_n$ , we have

$$pD_{2n} \equiv \begin{cases} -1 & (\text{mod } p) & (p-1 \mid 2n) \\ 0 & (\text{mod } p) & (p-1 \nmid 2n) \end{cases},$$

where  $p$  is any odd prime. Thus the denominator of  $\Delta_n$  can be determined (see below). The numerators of  $\Delta_n$  for  $0 \geq n \geq 4$  are all powers of 2. However this is really too good to be true; indeed we find that

$$\Delta_5 = -\frac{2^{15}}{3 \cdot 7 \cdot 11}, \quad \Delta_6 = \frac{2^{17} \cdot 191}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13}.$$

It may be of interest to determine the highest power of 2 dividing  $\Delta_n$ . A result of this sort is analogous to Kummer's congruences for the Bernoulli or Euler numbers [2, Chapter 14]; however the standard results on Kummer's congruences are restricted to odd moduli.

It is convenient to first transform  $\Delta_n$ . We have, using the symbolic notation,

$$\Delta_n = (D^2 - 1)^n = ((2B + 1)^2 - 1)^n = (4B^2 + 4B)^n = 4^n (B^2 + B)^n$$

and therefore

$$\Delta_n = 2^{2n} \sum_{r=0}^n \binom{n}{r} B_{n+r}. \quad (4)$$

Since  $2B_{2r} \equiv 1 \pmod{2}$  it follows readily that the denominator of

$$\sum_{r=0}^n \binom{n}{r} B_{n+r}$$

(in reduced form) is odd. Consequently

$$\Delta_n \equiv 0 \pmod{2^{2n}}. \quad (5)$$

Incidentally (4) is a bit more convenient for computation than (2). For example

$$\begin{aligned} \Delta_5 &= 2^{10} (5B_6 + 10B_8 + B_{10}) = -\frac{2^{15}}{3 \cdot 7 \cdot 11}, \\ \Delta_6 &= 2^{12} (B_6 + 15B_8 + 15B_{10} + B_{12}) \\ &= 2^{12} \left( \frac{1}{42} - \frac{15}{30} + \frac{75}{66} - \frac{691}{2730} \right) = \frac{2^{17} \cdot 191}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13}. \end{aligned}$$

The congruence (5) can be improved. We make use of the explicit formula

$$B_n = \sum_{r=0}^n \frac{1}{r+1} \sum_{s=0}^r (-1)^s \binom{r}{s} s^n,$$

which yields

$$(B^2 + B)^n = 2^n \sum_{r=0}^{2n} \frac{1}{r+1} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{1}{2}^s (s+1)\right)^n. \quad (6)$$

However allowance must be made for the denominator  $r + 1$ . Clearly the highest power of 2 contained in  $r + 1$ , where  $0 \leq r \leq 2n$ , is at most  $2^e$ , where  $e$  is determined by

$$2^e \leq 2n < 2^{e+1}. \quad (7)$$

Hence by (4) and (6),

$$\Delta_n \equiv 0 \pmod{2^{3n-e}}. \quad (8)$$

This result also can be improved for  $n \geq 2$ . Suppose first that  $n$  is even and let

$$2^k \mid n, 2^{k+1} \nmid n \quad (k \geq 1). \quad (9)$$

Then for arbitrary odd  $u$  we have

$$u^n \equiv 1 \pmod{2^{k+2}}. \quad (10)$$

Put

$$S_j = S_j^{(r)} = \sum_{s \equiv j \pmod{4}} \binom{r}{s} \quad (0 \leq j \leq 3).$$

It is easily verified that

$$2(S_1 - S_2) = (1 + i)^{r-1} + (1 - i)^{r-1}.$$

Hence for  $r$  odd

$$S_1 - S_2 = \begin{cases} 0 & (r \equiv 3 \pmod{4}) \\ (-1)^{1/4(r-1)} 2^{1/2(r-1)} & (r \equiv 1 \pmod{4}) \end{cases}.$$

Since  $s(s+1)/2$  is odd if and only if  $s \equiv 1$  or  $2 \pmod{4}$ , it follows from (10) that ( $n > 2$ )

$$S = \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{1}{2}^s (s+1)\right)^n \equiv S_2 - S_1 \pmod{2^{k+2}},$$

and therefore

$$S \equiv 0 \pmod{2^{k+2}} \quad (r \equiv 3 \pmod{4}), \quad (11)$$

$$S \equiv 0 \pmod{(2^{k+2}, 2^{1/2(r-1)})} \quad (r \equiv 1 \pmod{4}). \quad (12)$$

Now when  $r \equiv 1 \pmod{4}$ ,  $r + 1$  is divisible by 2 but not by  $2^2$ ; hence the term  $S/(r + 1)$  is integral  $\pmod{2}$  provided

$r > 1$ . When  $r \equiv 3 \pmod{4}$ , let  $2^{e_r}$  denote the highest power of 2 dividing  $r + 1$ . Then it follows from (11) that  $S/(r + 1)$  is of the form  $2^{k+2-e_r}A$ , where  $A$  is integral (mod 2). The least favorable case arises when  $e_r$  has its maximum value  $e$  as determined by (7). We have therefore

$$\Delta_n \equiv 0 \pmod{2^{3n-e+k+2}} \quad (13)$$

provided  $k + 2 > e$ ; when  $k + 2 \leq e$  we have

$$\Delta_n \equiv 0 \pmod{2^{3n-1}} \quad (14)$$

since for example the term  $r = 1$  has denominator 2.

In the next place for  $n$  odd, let

$$2^k \mid n - 1, \quad 2^{k+1} \nmid n - 1 \quad (k \geq 1);$$

Then (10) is replaced by

$$u^{n-1} \equiv 1 \pmod{2^{k+2}},$$

where  $u$  is odd. Put

$$\begin{aligned} T_j &= T_j^{(r)} = \frac{1}{2} \sum_{s \equiv j \pmod{4}} \binom{r}{s} s(s+1) \\ &= \binom{r}{2} S_{j-2}^{(r-2)} + r S_{j-1}^{(r-1)}. \end{aligned}$$

Thus

$$\begin{aligned} T_1 - T_2 &= \binom{r}{2} (S_3^{(r-2)} - S_0^{(r-2)}) + r (S_0^{(r-1)} - S_0^{(r-1)}) \\ &= \frac{i}{4} \binom{r}{2} \{(1+i)^{r-1} - (1-i)^{r-1}\} + \frac{r}{4} \{(1+i)^r + (1-i)^r\}. \end{aligned}$$

Simplifying we get (for  $r \geq 3$ )

$$T_1 - T_2 = \begin{cases} \frac{1}{4} r (r+1) 2^{\frac{1}{2}(r-1)} i^{\frac{1}{2}(r+1)} & (r \equiv 3 \pmod{4}) \\ \frac{1}{2} r (2i)^{\frac{1}{2}(r-1)} & (r \equiv 1 \pmod{4}). \end{cases}$$

Also ( $n > 3$ )

$$T = \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{1}{2} s (s+1)\right)^n \equiv T_2 - T_1 \pmod{2^{k+2}}$$

and therefore in particular

$$T \equiv 0 \pmod{(2^{\frac{1}{2}(r-1)}, 2^{k+2})} \quad (r \equiv 3 \pmod{4}). \quad (15)$$

As in the case  $n$  even, it will suffice to take  $r \equiv 3 \pmod{4}$ . Let  $2^{e_r}$  denote the highest power of 2 dividing  $r + 1$ , and consider  $2^{-e_r} T$ . For  $e_r \leq 2$ , we get the exponent  $-1$ ; for  $e_r \geq 3$  it follows that  $\frac{1}{2}(r - 1) \geq e_r$ . Hence if  $k + 2 < e_r$ , we get the exponent  $k + 2 - e_r$ , while if  $k + 2 \geq e_r$  we again get  $-1$  (at most). Consequently

$$\Delta_n \equiv 0 \pmod{2^{3n-e+k+2}} \quad (k + 2 < e_r) , \tag{16}$$

$$\Delta_n \equiv 0 \pmod{2^{3n-1}} \quad (k + 2 \geq e_r) . \tag{17}$$

Comparing (16) and (17) with (13) and (14) we may accordingly state the following

**THEOREM.** *Let  $2^e \leq 2n < 2^{e+1}$  and let  $2^k$  denote the highest power of 2 dividing  $n$  or  $n - 1$  according as  $n$  is even or odd. Then  $\Delta_n$  as defined by (1) and (3) satisfies (16) and (17). Applying the Staudt-Clausen theorem, it follows from (3)*

that

$$p\Delta_n \equiv - \sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \pmod{p} ,$$

where  $p = 2m + 1$  is an odd prime. It can be shown that

$$\sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \equiv \begin{cases} 2 & (p - 1 \mid n) \\ \binom{m}{k} & (p - 1 \mid n + k, 0 < k \leq m) \\ 0 & \text{otherwise.} \end{cases}$$

In particular the prime factors of the denominator of  $\Delta_n$  are simple and cannot exceed  $2n + 1$ .

In connection with formula (4) above it may be of interest to cite the formula [2, p. 189]

$$(-1)^n \sum_{r=0}^n \binom{n}{r} \frac{B_{n+r+1}}{n+r+1} + \frac{n! n!}{2(2n+1)!} = 0 .$$

#### REFERENCES

1. G. FROBENIUS, *Über die Bernoullischen Zahlen und die Eulerschen Polynome*, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (1910), pp. 809-847.
2. N. NIELSEN, *Traité élémentaire des nombres de Bernoulli*. Paris, 1923.
3. N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung*. Berlin, 1924.  
Duke University.