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SCIENTIFIC FOUNDATIONS OF SCHOOL MATHEMATICS

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Set and *function* (transformation, mapping, association, etc.) are basic notions in mathematics (and not only in Mathematics). It is amazing to observe that one tends to conceive the notion of set too generally and that of function too restrictively. Both these notions should be conceived quite generally. It is of vital importance that the notion of function also be understood very generally to embrace the processes of associating one thing with another, in particular to embrace geometrical transformations, physical mappings and various associations in everyday considerations: the transition from particular to general and conversely must be an everyday procedure.

One has to *differentiate* a *function* from the *expression* of the same function. A function may be expressed in very many ways, even in many analytical ways; a function can be expressed or visualized in qualitatively different ways (analytically, mechanically, geometrically, optically, electrically, biologically, economically, etc. etc.).

1. *Set considerations* consist in examining the constituent parts, in particular how the whole is composed of its elements or points and of some less or more characteristic parts. Obviously, the set considerations are not exhaustive, but do contribute to better understanding of many things. In particular, it is necessary to ask what are the constituent parts of a thing S and how they are transformed by a mapping of S. Formally, given S we ask for the solutions x of $x \in S$ and the solutions of $x \subseteq S$.

1.2. *The union* (the U-operator) and *intersection* of sets (the $\cap$-operator) are to be used everywhere. In particular, a straight line AB is to be defined as the union of all intervals containing each of the points A, B. The triangle ABC is to be defined as the union of intervals whose extremities lie in the union $AB \cup BC \cup CA$. The plane ABC is the union of all circles or triangles containing the points A, B, C, etc.

2. *Function (functor, mapping, transformation, association...).*

2.1. Function of a class M into a class M' means that to every element of M one associates one or more elements of M'. If f denotes such a function from M to M' then for every $x \in M$ one denotes by fx or xf the corresponding " value " or each of the corresponding values that are associated with x.

For a given $x \in M$ one denotes by $\{ fx \}$ or $\{ xf \}$ the class of all the values fx. One speaks of a *set-function* or *S-function or relation* $x \rightarrow \{ fx \}$ meaning that to every element in M one associates a set. In a general case fx might mean very different items (e.g. points, sets, phenomena, etc.). E.G. if, M denotes the inhabitants of a town, fx might mean the main name of x. In particular cases when M means a geometrical or arithmetical set, fx might mean a geometrical object or some other strictly defined mathematical topic: number(s), figure(s), set(s) etc.

2.2. *Domain and antidomain.* If f is a function from M to M', M is called the *domain of f* and might be denoted Df. The union of all the $\{ fx \}$, x running over Df, is called the *anti-domain or range* of f and might be denoted ${}^{-}Df$ (the sign ${}^{-}$ is placed on the top at left and is read: *anti* or *counter*). For any non null $X \subseteq M$ one defines fX as the union of all the $\{ fx \}$, for $x \in X$. Thus $fM = {}^{-}Df$.

2.3. *Equality of functions.* The functions f, g are equal: $f = g$, provided both $Df = Dg$ and $fx = gx$ for every $x \in Df$. If not $f = g$ then $f \neq g$.

2.4. *The antifunction ${}^{-}f$ or f^{-1}* of f is defined in ${}^{-}Df$ so that for every $y \in {}^{-}Df$ ${}^{-}fy$ denotes the point(s) $x \in Df$ satisfying $fx = y$.

Consequently for every function f in a domain D we have to consider Df and ${}^{-}Df$ as well as the antifunction ${}^{-}f$ and its domain $D^{-}f$ and its antidomain $^{-}D^{-}f$. One proves readily that

2.5. $Df = {}^{-}D^{-}f$, $^{-}Df = D^{-}f$, $^{--}f = f$.

3. *Relation.* The notion of a relation is very general. If M and M' are classes then any part A of the product $M \times M'$ gives rise to a (binary) *relation* (or is a relation) A on M to M'. Any function f from M to M' gives rise to a relation: $x \rightarrow \{fx\}$. Moreover one speaks of relations in connexion with *particular* elements too. For instance, the equality $2 + 3 = 5$ is a ternary relation; by passing from particular to general one gets the corresponding function $x + y = z$ or $z = x + y$. Consequently, one aspect of the connexion between relations and functions is the connexion between the logical *quantors* (or *quantifiers*): the *particular* and *general*, *some* and *every* (each).

4. *Logical quantors* or *quantifiers.* An essential aspect of the foundation of school mathematics consists in a rational and constant application of the logical quantifiers or quantors: *some* and *every* (or equivalent expressions: *at least one*, *every*, *all*, *without exception*, etc.). Set and function are very appropriate topics for using the quantifiers. Conscious use of *quantors* and of *logical constants (and*, *not*, *or*, *implies)* is one of the characteristic aspects of modern mathematics which tend to form a unity with logic.

5. *Inverse processes or antiprocesses.* To every function, functor, relation, process is associated an inverse one. Both processes are to be considered. An aspect of this idea consists in examining whether the converse of a statement holds.

6. *Cloud functions.* In our opinion it is of primary importance to consider not only uniform functions and relations but general ones too. In particular, every plane set S is to be considered as the graph of a function, the domain and the antidomain of which are the first and the second projection of S. Every " *cloud of points* " in R^2 (= plane) i.e. every arbitrary (finite) set of points (x, y) is a representation of a determined function; such " *cloud functions* " are vital for statistical considerations and applications of mathematics in biology, industry, economics and human sciences etc. Nowadays such " monster-

functions " are banished not only from school mathematics but are not considered as functions at all. The great idea of functional thinking covers very little if the notion of Function is taken in a too restrictive sense. In order to save the Functional Standpoint, it is necessary to allow that Function covers every type of " functions " occuring in everyday human activity, industry, economics, technology, sciences, etc. For instance, languages are a paramount example of functional considerations, with sentences as variables, the connectives being the functors. The stochastic functions and models enable wider considerations and applications to be included.

An exemple in this connection and concerning the uniformization of a given function reads as follows. The function $f \ldots y = x^{\frac{1}{2}}$ with R_0 (= set of all real numbers $\geqslant 0$) as domain is uniform only in O. There are many uniform functions g with R_0 as domain and satisfying $gx \in \{fx\}$. Such are the functions $|x^{\frac{1}{2}}|$, $-|x^{\frac{1}{2}}|$, $(-1)^{\mathrm{Ex}}|x^{\frac{1}{2}}|$ etc. There are 2^c of such uniformizing (or stochastic) functions with respect to $y = x^{\frac{1}{2}}$; only two are continuous viz. $|x^{\frac{1}{2}}|$ and $-|x^{\frac{1}{2}}|$. For instance, if $h(x)$ is any uniform mapping of R_0 into $\{0, 1\}$, then $(-1)^{h(x)}|x^{\frac{1}{2}}|$ is such a function attached to $x^{\frac{1}{2}}$.

6.1. It is extremely interesting to compare a given set as a representation of a function (e.g. the parabola $y = x^{\frac{1}{2}}$ in the preceding case or a " cloud " of points in R^2) and some uniform " classical functions " associated with it. For instance, if the given " cloud " consists of 100 points, there are in general $\binom{100}{2}$ straight lines attached to it but only one that fits *collectively* to the cloud in the sense of regression line of y in respect to x (linear regression in statistics, in connection with the method of least squares). Classical functional standpoint is useful as one approach to the question but does not exhaust the question.

6.2. Patterns for mathematical investigations are not only macroscopic physical phenomena but also biological, economic, human, atomistic phenomena, where the classical strict functional relations do not necessarily hold. Therefore the idea of organization or structure is certainly fundamental for modern mathematics.

7. *Structure or organization.* Vaguely speaking, a structure or organization is any whole with certain interrelation(s) between its parts. There are some pilote-structures inside and outside mathematics.

7.1. Here are some structures outside mathematics: a state, human body, traffic system (in a city, state or all over the world), languages, machines...

7.2. In mathematics we are dealing with several structures, the simplest one being the possibility of *equalling* and *distinguishing* given things (relations $=$ and $\neq$).

7.2.1. *Orders relation* with the inclusion relation $\subseteq$ as pattern is the next fundamental example of mathematical structure. It is interesting to observe that the divisibility is an order relation in the set of natural numbers.

7.2.2. To every set S we associate its square S^{I2}, cube S^{I3} and the S^{In} for $n > 3$. Mappings of these sets into the initial set S are of great importance. In particular, every mapping f of S^{I2} into S is interpreted as a *groupoid* structure $(S; f)$ of S. This idea covers what really is basic in all kinds of calculations. The main idea is that given a and b in S the result $f(a, b)$ or afb (say $a + b$, $a - b$, etc.) lies again in S, provided $(S; f)$ be a groupoid. There are several kinds of groupoids *(semigroups, groups, etc.).* In the simplest cases the groupoid structure $(S; f)$ enables us to map the spaces S^{I2}, S^{I3}, ... S^{In} into S and one into another more generally. For instance, R being the set of real numbers $(R; +)$ is a groupoid and the $+$ — operation enables us to map R^{I2}, R^{I3}, ... and even (partly) $R^{I\omega}$ into R.

7.2.3. A further main mathematical structure consists of any mapping f of PS into PS (PS denotes the set of all the parts of S). Every mapping f of PS into PS is called a space whose points are those of S; it is denoted $(S; f)$ and in particular $(S; {}^{-})$. For instance, if to every $X \subseteq R$ we associate the set $\overline{X}$ of all points $a \in R$, each of which is a limit point of some sequence extracted from X, the set R becomes a space—the space $(R; {}^{-})$ of real numbers.

7.3. The previous examples of structures show how the idea of a structure is a synthesis of the ideas of set and function.

Everywhere (in mathematics and in other fields) we are dealing with structures. It is of fundamental importance to recognize the polyvalency of structures. In classical mathematics we are too much accustomed to deal with some kinds of numbers, stopping usually at complex numbers. Now, given e.g. any set S, the set $S\,!$ of all the permutations of S yields a structure, a group with respect to composition of permutations.

7.4. The *polyvalency of structures* has another aspect, expressible in the form of the following axiom:

Every phenomenon is a generator of a (mathematical or logical) structure obtained by the transition from particular to general. One of the fundamental tasks is to find where a given structure occurs and with what degree of fidelity it occurs (possibility of various approximations, deviations and nuances of a structure).

7.5. From the structural point of view there is no difference between arithmetical, analytical or geometrical considerations; they are various aspects of one structure.

Consequently, the idea of structure implies a fusion of many aspects, many theories or domains that historically have existed independently until now.

8. We consider as *fundamental acquisitions* the following topics:

1. *Implication* and *quantors:* some and every, and connected items. The classical question: "*How much or how many ?*" is to be answered now not only by 1, 2, 3, ... but also by these items:

   some and *every* (or all). We contribute to a unification of mathematics and logic.

2. *Groupoidal and closeness considerations:* whether the output or result lie in the system in which are the inputs or data.
3. *Order considerations*, one aspect of which are *qualitative* and *extremality considerations*, respectively; the processes usually run in such a way that some minimum (maximum) condition is satisfied.

4. The ideas of set and function must be general and used everywhere in mathematics. They are linked together in the basic notion of a structure or organization. It is only in this way that the ecological standpoint may be applied everywhere and embrace in particular various statistical considerations and applications.

5. The domain of mathematics, its applications and interrelations with other topics is much more extensive than earlier; therefore the usual terms (variables, function, set, etc.) are to be used in a much more general sense than previously.

9. The preceding items should be, in our opinion, the very scientific basis of school mathematics. In this connexion I am anxious to say that many of these items are being introduced and taught in Yugoslavia—at least in some of its six republics. In this respect the Yugoslav subcommittee of the I.C.M.I., the Union of Societies of Mathematicians and Physicists of Yugoslavia and the republican societies of mathematicians and physicists were very active. We organised special seminars and meetings (at least one a year) dealing with specific topics e.g. Statistics and probability, Sets, Calculus, Vectors, Numerical analysis, Interrelations of mathematics and physics... The results we attained so far are very satisfying.

For some details, see also:

1. G. Kurepa, The role of mathematics and mathematician at present time (International Inquiry of the I.C.M.I.). *Proc. Int. Congress of Mathematicians*, Amsterdam, 1944 (vol. 3 (195), 305-317.
2. —— Sur les principes de l'enseignement mathématique. *L'Enseignement mathématique*, Genève, accepted for publication.