MEMORIAL TRIBUTE TO J. H. C. WHITEHEAD

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MEMORIAL TRIBUTE TO J. H. C. WHITEHEAD [1])

by P. J. Hilton

It would, in any circumstances, be a great honour to be invited to give an address at this colloquium. But the particular circumstances which constitute the occasion for my address to you today are so deeply significant for us all that I find myself much more than usually aware of the responsibility that rests upon me. Henry Whitehead was one of the greatest English mathematicians; and I, as his pupil and close friend, am grateful for this unique opportunity to pay my tribute to his memory.[2]) I must apologise in advance that the time available to me to prepare this address has permitted me to do little more than describe in outline his life and work. However, I must in all honesty admit that a considered estimate of the significance of Whitehead's many contributions to mathematics would not only require immensely more time for preparation than the exigencies of the English university examination system have allowed me; it would also require that I possess the breadth and depth that Whitehead himself showed. Whitehead wrote some 85 papers, covering about 1500 pages, and few of those papers may be neglected in making the final judgment.

John Henry Constantine Whitehead was born in India on November 11th, 1904. His father was the Rt. Rev. Henry Whitehead, Bishop of Madras, and his uncle was A. N. Whitehead, the philosopher. He was educated at Eton College and at Balliol College, Oxford. After graduating he went into the City, making and losing a fortune with an enthusiasm and gaiety which we who knew him can easily imagine. The decisive moment in his life came in 1928 with his return to academic work.

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[1]) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, Wednesday, June 22nd, 1960.

[2]) I would like to take this opportunity to express my appreciation to those colleagues who have helped me to prepare this tribute.

He won a Commonwealth Fund Fellowship in 1929 and went to Princeton for 3 years to study geometry under Veblen. His Ph. D. thesis at Princeton was entitled *The Representation of Projective Spaces* (Princeton, 1931); and after completing it he collaborated with Veblen in writing the Cambridge tract, *Foundations of Differential Geometry*, which is now part of the classic literature. On returning to England he became a Fellow and Tutor of Balliol College, a position he held until he left Oxford to take up war service in 1940. He married in 1934, a marriage which enriched his life and the lives of all those privileged to know " Henry and Barbara ", a marriage whose happiness was reflected in his work. During the period 1933-41 he embarked on his fundamental researches in combinatorial topology and published many papers of great originality whose profound significance is only now being elucidated. In 1941 he was engaged on work for the Admiralty and in 1943 transferred to the Foreign Office; I may be forgiven for stressing this last date for it was then that we met, since I was already engaged on Foreign Office work. You will readily believe that this period during which *I* was teaching Henry Whitehead was of brief duration; but the friendship begun in 1943 led naturally to my becoming *his* pupil in 1946 on my release from war service and thus was decisive in fashioning my life.

Henry Whitehead was elected a Fellow of the Royal Society in 1944, while still on war service. In a characteristic jest he described himself as the last of the real F.R.S.'s as it was known that the annual number elected was to be raised from 16 to 25 in 1945 !—a joke typical of an essentially modest man. In 1945 he was chosen to succeed Dixon as Waynflete Professor of Pure Mathematics in the University of Oxford, a position carrying with it a fellowship of Magdalen College. He attracted a great number of research students to study under him in Oxford and mathematicians came from many other countries to work with him. His research output was prodigious and his volume of publications was further swollen by several papers in which he set out deliberately to recast some of his earlier work in a form more easily assimilated by his colleagues. For, to Henry Whitehead, the writing of a paper was an act of communication and he was very sensitive to the reputation for obscurity which he believed he possessed. He devoted himself in this period largely to algebraic homotopy theory and, as I have indicated, established Oxford as one of the leading centres of mathematical activity in this and related fields. Fortunately the presence of Ioan James in Oxford ensures that the tradition he established there will not die with him.

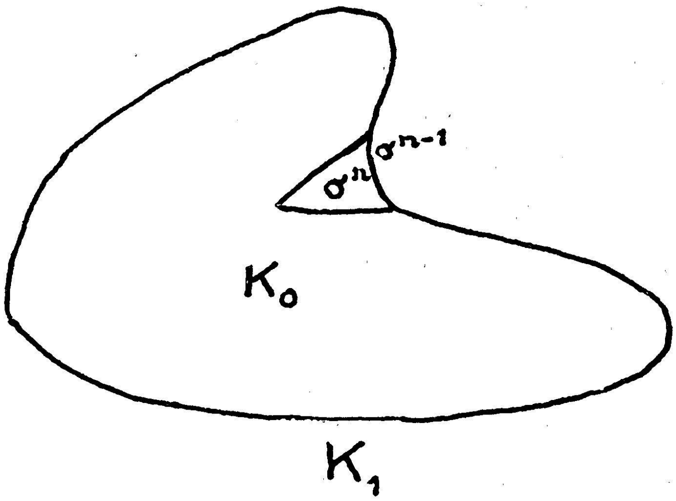
During the last few years his interests began to turn again to the problems he had studied before the war; — this is reflected in the title, *On 3-dimensional manifolds*, of the address he was to have delivered to this colloquium; and his boyish gratification at the interest shown by others in this early work of his was a delight to see. The proof by Papakyriakopoulos of Dehn's Lemma brought him intense joy and stimulated him to renew his work on manifolds. His enthusiasm was so great, and his physical and intellectual energy so formidable, that he could only do justice to himself by taking a year's leave of absence from his duties in Oxford to devote himself to his research and to be free to discuss common problems with his fellow-mathematicians. He was spending this year's leave in the U.S.A. fruitfully and happily when on the morning of Sunday, May 8th, in Princeton, he collapsed and died.

I have already disclaimed both the intention and the competence to deal comprehensively with Whitehead's many contributions to mathematics. Of his earliest work in differential geometry I will say nothing; and a passing mention must suffice of his work in pure algebra. In his paper [1] he gave a general proof of the Levi theorem that the radical of a Lie algebra over a field K of characteristic zero is complemented. In effect he proved that the first and second cohomology groups of a semi-simple Lie algebra L over K with coefficients in an L-module vanish (Whitehead's first and second lemma) and in a subsequent paper [2] deduced from the first lemma Weyl's theorem that the representations of a semi-simple Lie algebra L over K are completely reducible. He applied the same techniques to associative algebras in his paper [3]. Back in the 1930's Whitehead also addressed himself to the study of free groups [4, 5]. However in these two papers, whose titles sufficiently indicate the problem studied, the methods and, in part, the motivation belonged to combinatorial topology. Undoubtedly Whitehead's greatest work before the war lay in the field of combinatorial topology and of this work, and of its later developments, I will now say something.

Whitehead met Newman in Princeton in 1931 and they became close and life-long friends; and it must have been largely under Newman's inspiration that Whitehead became passionately interested in combinatorial topology. Whitehead devoted a great amount of effort and time to trying to prove the Poincaré conjecture (he published a " proof " in 1934 which he immediately withdrew). He never regarded the time spent on this problem as wasted. In 1935 he published a paper [6] describing a semilinear open simply-connected 3-dimensional manifold with vanishing second homology group which was not in (1,1) semi-linear correspondence with Euclidean 3-space. Subsequently in a joint paper with Newman [7] it was shown that this manifold was not even topologically a 3-cell, thus disposing of the Poincaré conjecture for open 3-manifolds. Whitehead showed his remarkable insight and resilience when he returned with renewed energy to the Poincaré conjecture after Papakyriakopoulos had at last established the validity of Dehn's Lemma and had proved the sphere theorem, though under an uncomfortable additional hypothesis. In two remarkable papers [8, 9] he first modified *P*'s proof of the sphere theorem in such a way as to get rid of the restrictive hypothesis and then, in collaboration with Arnold Shapiro, he so simplified the proof of Dehn's Lemma (by using 2-sheeted instead of universal coverings) that it is now " available to all ". Essentially we have a Dehn curve C on the boundary of a compact 3-manifold [1]) V and bounding a Dehn disc D. C is good if it bounds a non-singular disc. Then Shapiro-Whitehead show, first, that if V has no 2-sheeted covering C is good; second, that if $p\colon V_1 \to V$ is a 2-sheeted covering and $p^{-1} C = C_1 \cup \tau C_1$ where C_1 is a Dehn curve above C, τ the cover transformation, then if C_1 is good so is C (this is essentially the valid part of Dehn's original

1) If D lies in M, then V is a regular neighbourhood of D in M (see below).

argument); and third that if D lifts canonically to D_1 then D_1 has fewer closed double curves than D. I would like to mention, in connection with the work of Whitehead on the sphere theorem, that D. G. A. Epstein has recently generalized this theorem to non-orientable manifolds. Thus Whitehead's version of the sphere theorem asserts that if M is a connected orientable triangulated 3-manifold embedded in a space X and if there is a map $S^2 \to M$ which is essential in X then there is a non-singular



$K_0 = K_1 - Int\, \sigma^n - Int\, \sigma^{n-1}$

polyhedral 2-sphere in M which is essential in X; Whitehead deduced from this that there exist a finite number of disjoint non-singular polyhedral 2-spheres in M whose transforms by elements of $\pi_1(M)$ generate $\pi_2(M)$. Epstein discards orientability and proves that, under the remaining hypotheses of the sphere theorem, there is a non-singular 2-sphere or projective plane in M which is essential in X and further that there exist a finite number of disjoint non-singular polyhedral 2-spheres or projective planes in M whose transforms generate $\pi_2(M)$.

Newman's fundamental work in combinatorial topology in the 1920's and 1930's again showed its influence in one of Whitehead's most important papers [10]. This paper contains the germs of many ideas which are now familiar to algebraic topologists but which were then quite new. The distinction between simplicial spaces and nuclei is essentially the difference between combinatorial moves in the sense of Alexander and in the later sense of Newman. Whitehead defined a *nucleus* as an equivalence class of (finite) complexes under formal deformations. An *elementary contraction* is a process $K_1 \to K_0$ and an *elementary expansion* its inverse; then a *formal deformation* is a finite sequence of elementary expansions or contractions. Certainly K and a subdivision K' of K have the same nucleus so that the nucleus is a combinatorial invariant—it is unknown whether it is a topological invariant. The nucleus is now called the *simple homotopy type* of a complex and it is indeed true that complexes of the same simple homotopy type have the same homotopy type. The *m-group* of a complex is defined by means of elementary operations of filling of order k and perforating of order k. *A perforation of order k* is the removal of the interior of a principal k-simplex, a filling its inverse. Then K and L have the same m-group if L is obtainable from K by a finite sequence of fillings and perforations of order exceeding m. Whitehead chose the name " m-group " because the 2-group of a connected complex is completely determined by its fundamental group; however in his post-war papers he changed the name to " m-type " and more recently, in response to popular request, the name has been changed to " $(m - 1)$-type ". I will adopt the latter designation so that $(m - 1)$-types are determined by perforations and fillings of orders greater than m. It is not hard to see that complexes of the same homotopy type have the same n-type for every n; conversely, two (finite) complexes are of the same homotopy type if they are of the same n-type for every n. A deeper result is that if K, L are n-dimensional and have the same $(n - 1)$-type then clusters of n-spheres can be attached at single points of K and L so that the resulting complexes have the same simple homotopy type.

A further basic notion introduced in this paper is that of a *regular neighbourhood* of a complex K embedded in a combinatorial manifold M, of dimension n, say. This is a subcomplex N of M such that N is itself an n-manifold and N contracts geometrically into K (i.e. we pass from N to K by discarding q-cells hitched on by $(q - 1)$-cells on their boundary). Whitehead proved that if $K \subseteq M$ then K has a regular neighbourhood; namely, if $s_K M$ stands for the subdivision of M obtained by starring the simplexes of $M - K$ in order of decreasing dimensionality and if $N(P, Q)$, for P a subcomplex of Q, is the set of closed simplexes in Q which meet P, we may take $N = N(K, s_K^2 M)$. Whitehead also proved that any two regular neighbourhoods of K in M are combinatorially equivalent, and that if two complexes embedded in Euclidean space R^p for p sufficiently large are of the same simple homotopy type their regular neighbourhoods are combinatorially equivalent. The consequences that flow from the precise forms of these concepts and results are numerous—and certainly not yet exhausted. The striking theorem proved in a paper quoted below that if M^n is a contractible smooth manifold then $M^n \times E^{n+5} = E^{2n+5}$ was a particular favourite of mine. More recently Zeeman has made essential use of the fact that a regular neighbourhood of a geometrically collapsible K in an n-manifold M is an n-element in his proof that S^n cannot be knotted in Euclidean R^k if $k > \frac{3}{2}(n+1)$ [1]; and Penrose, Zeeman and Whitehead have used the same result to prove some very interesting theorems on embedding manifolds in Euclidean space, among them that a $(k-1)$-connected closed combinatorial n-manifold, $0 < 2k \leq n$ may be rectilinearly embedded in R^{2n-k+1}.

Whitehead further developed the ideas of the 1939 paper [10] in two consecutive papers in the *Annals* [11, 12]. A C^1-complex, $f(K)$, is the image of a map $f: K \to R^n$ which is of class C^1 on each simplex σ of K (i.e., $f \mid \sigma^k$ extends to a C^1-map of an open neighbourhood U of σ^k in R^k) and two C^1-triangulations of a manifold are combinatorially equivalent. Among recent applications of this notion I might instance Thom's definition of combinatorial Pontryagin classes. In the second paper, Whitehead pointed out a consequence of his earlier results, namely, that, for manifolds M_i^n, $i = 1, 2$, belonging to a certain class Π, if M_1^n and M_2^n have the same simple homotopy type then $M_1^n \times \sigma^k$ is combinatorially equivalent to $M_2^n \times \sigma^k$ for k sufficiently large. The manifold $M^n \in \Pi$ if, for k large, its regular neighbourhood in R^{n+k} is equivalent to $M^n \times \sigma^k$ and Whitehead showed that

[1] Added in proof: Zeeman has recently shown that S^n can only be knotted in R^{n+2}.

$M^n \in \Pi$ if its normal bundle in R^{n+k} has a cross section, for some k. The relationship of this work to the generalized Poincaré hypothesis is very close.

In these papers Whitehead's approach was almost entirely combinatorial, although in the 1939 paper he did discuss certain algebraic notions, including the conditions on the fundamental group of K which would ensure that complexes of the same homotopy type as K would be of the same simple homotopy type. He translated the combinatorial moves into algebraic transformations in a paper [13]; there he defined the *natural system* of a complex K as a presentation of $\pi_1(K)$ together with the incidence matrices in $\tilde{K}$. Elementary algebraic transformations of natural systems give rise to the notions of L-equivalence, corresponding to simple homotopy type, and L^*-equivalence, corresponding to homotopy type. One should mention here the influence of Reidemeister's work in connection with the matrix transformations concerned in L-equivalence. The L-transformations also include Tietze transformations on presentations of $\pi_1(K)$ and Whitehead's theorem on attaching clusters of spheres to n-complexes of the same $(n-1)$-type is now seen as a generalization of the Tietze theorem on the relation between presentations of a group. It is worth recording that it was in this paper that Whitehead solved the homotopy classification problem for lens spaces. Whitehead had always been fascinated by Hurewicz' question whether there exist non-homeomorphic n-manifolds of the same homotopy type; we know now, by virtue of Whitehead's classification and the Moise theorem that the lens spaces, e.g., (7.1) and (7.2) provide examples.

The final paper which I want to mention in connection with this phase of Whitehead's work in fact appeared very much later, and was part of his programme of presenting his ideas in a more algebraic form or, rather, of displaying the algebraic aspect in order that the ideas should be better understood. The paper I refer to is [14]. If, as we are asked to accept, January, 1960 marks the beginning of a new decade, so also surely did this paper of Whitehead's. But it was in a sense, too, a rounding-off of Whitehead's work and it was not until quite recently that he, and others, returned to it as a source of fresh ideas. In his new approach to simple homotopy types, Whitehead considered free π-complexes and defined the *torsion* τ of a chain-equivalence $f: C \simeq C'$ of such complexes. A simple equivalence is then a chain-equivalence f for which $\tau(f) = 0$. Now a homotopy equivalence ϕ: $K \rightarrow L$ gives rise to a chain-equivalence $f: C(\tilde{K}) \rightarrow C(\tilde{L})$ where $C(\tilde{K})$ and $C(\tilde{L})$ may both be regarded as free $\pi_1(K)$-modules through the isomorphism $\phi_*: \pi_1(K) \cong \pi_1(L)$. Whitehead showed that ϕ is a simple homotopy type equivalence if and only if f is a simple equivalence.

I have spoken at some length of this work of Whitehead's because I believe it to be of absolutely fundamental importance and because I believe it reveals Whitehead's great powers as a mathematician. A colleague of mine—and also a pupil of Whitehead's—has put the matter excellently in saying that this work " gives a clear picture of how much Whitehead was a part of, and a maker of, contemporary tradition in algebraic topology ". He brought geometry, combinatorics and algebra together, using each to complement the others, accepting the greatest degree of abstraction where appropriate but always returning finally to the geometrical source to give substance to his results.

I come now to Whitehead's contributions to algebraic homotopy theory in the period from 1946. Whitehead was always an originator and innovator; but as we have seen already he was also strikingly good at understanding the significance of the work of others, adapting their ideas to his purposes, developing and deepening their results, and sometimes indeed greatly clarifying them in the process. It is thus difficult to select from so varied an array of mathematical work, but I think that perhaps his most significant contributions in this period are in the development of combinatorial homotopy, in his repeated stress on the importance of realizability and, later, in his invention with Spanier of S-theory.

The notion of a CW-complex, the germ of which, along with that of other now standard notions of algebraic homotopy theory (the mapping cylinder, the exact homotopy sequence, killing homotopy groups by attaching cells, . . .) had already been planted in the 1939 paper, [10], was first introduced explicitly by Whitehead in an address to the Princeton meeting of the

A.M.S. in November, 1946. However the revised manuscript of the address was not received till July, 1948 and was not published until the following March. The address appeared in two parts [15], and was followed by the paper " Simple Homotopy Types " in the *Amer. J. Math.* to which reference has already been made. Whitehead described the purpose of those papers as that of clarifying the theory of nuclei and m-groups, but I think we would now agree that they did that and much more. First he gave the definition of a closure finite cell-complex with the weak topology, or CW-complex, and established its principal properties. There is no doubt that this has proved a most fruitful combinatorial concept and that it is often natural to regard the spaces under discussion as built up by the process of attaching cells or, equivalently, as decomposed into cells. The category of CW-complexes is closed under passage to covering spaces and the construction of mapping cylinders with respect to cellular maps; it is also closed under the taking of topological products provided the factors are countable but an example due to Dowker shows it not to be so in general. Further the singular homology groups may readily be deduced from the combinatorial structure. Among consequences of the facts quoted above are the important theorem that a map of CW-complexes which induces an isomorphism of homotopy groups is a homotopy equivalence; also a map which induces isomorphisms of the fundamental group and of the homology groups of the universal covers is a homotopy equivalence. There are corresponding theorems in which the notion of equivalence is replaced by n-equivalence; Whitehead gave a definition of n-type based on Fox's notion of n-homotopy type which he proved equivalent to the notion n-group defined in his 1939 paper in terms of elementary transformations. I think these theorems are very beautiful; but, beyond their aesthetic appeal, they have the immense value of drawing attention to the importance of realizability, that is, of exhibiting a geometrical map inducing given isomorphisms of homotopy or homology groups. How often in the literature does one meet the situation in which, wishing to study the homotopy groups of Y, one has a map $f\colon X \to Y$ where X, Y are 1-connected and one proves that f induces homology isomorphisms up to some dimension. Serre did justice when he described the inference that f induces homotopy isomorphisms up to (but excluding) the same dimension as Whitehead's theorem. Recently I have learnt of more than one instance of importance where it is known that the homology groups of X and Y are isomorphic and the problem is to find a map which induces the isomorphism or to prove that an obvious candidate map has the desired property. The important place rightly accorded to c.s.s. complexes as a combinatorial gadget in algebraic homotopy theory does not, I believe, require us to modify our attitude towards CW-complexes. In fact, John Milnor has recently published a strongly propagandist pamphlet for CW-complexes showing that the category of spaces of the homotopy type of CW-complexes is closed under the loop space functor; and Kan's recent penetrating work on group complexes has shown among many other striking results that the theory of CW-complexes is essentially equivalent to that of free group complexes.

Whitehead gave further emphasis to the notion of realizability when he showed, in a short but much-quoted paper [16] that every system consisting of a group π_1, and π_1-modules π_2, π_3, ... could be realized as the system of homotopy groups of a CW-complex. It remains an interesting question to decide what abstract Whitehead products $\pi_m \otimes \pi_n \rightarrow \pi_{m+n-1}$ may also be realized.

Certainly, however, not every isomorphism of homotopy or homology groups can be realized geometrically. Thus a more refined algebraic system must be associated with a complex in order that a complete system of homotopy invariants may be obtained. Whitehead addressed himself to this problem with striking success in his paper [17]. In this paper he defined the cohomology spectrum of such a polyhedron to be its set of cohomology rings $H^* (K; Z_m)$ for $m = 0, 2, 3, \ldots$ ($Z_0 = Z$), the coefficient homomorphisms $\mu_{r,s}$: $H^n (K; Z_r) \rightarrow H^n (K; Z_s)$, the Bockstein operations Δ_m: $H^n (K; Z_m) \rightarrow H^{n+1} (K; Z)$, and the Pontryagin squares p_{2r}: $H^n (K; Z_{2r}) \rightarrow H^{2n} (K; Z_{4r})$. These elements of structure satisfy certain basic relations. An (abstract) 4-dimensional cohomology " ring " is readily defined (it is a spectrum as above whose cohomology groups could be those of a simply-connected-4-dimensional polyhedron). Then Whitehead proved that every such " ring " could be realized as the spectrum of a simply-connected 4-dimensional polyhedron and every ring-homomorphism from the spectrum of L to that of K could be realized by a map $f\colon K \to L$. Then a theorem already quoted shows that f is a homotopy equivalence if f^* is an isomorphism. If one suspends such polyhedra one gets a simpler cohomology spectrum which characterizes the homotopy type in the stable range; in particular the Pontryagin squares are replaced by Steenrod squares. I know that the significance of these results was well understood at the time the paper appeared and Whitehead was very gratified by the plaudits he received, privately, from his colleagues. It is true that Whitehead's result has not led to a convenient general notion of a cohomology spectrum for arbitrary polyhedra nor to any simple normal form for $(n-1)$-connected polyhedra, though the direction of attack has proved well worth following. The search for a complete set of homotopy invariants has taken a different direction, in some sense a dual direction, and one is now conditioned to think of the Postnikov invariants in this connection. But we should comment that the cohomology spectrum is more easily apprehended and computed than the Postnikov system, for example, for a sphere. It should also be remembered that Whitehead himself contributed to the understanding of the role of the Postnikov invariants; his paper with MacLane [18], whose title should now be amended to " *On the 2-type of a complex* ", showed that an algebraic 2-type, consisting of a group π_1, a π_1-module π_2, and a cohomology class $k \in H^3(\pi_1; \pi_2)$ could be realized by a CW-complex K and determined the 2-type of K. Whitehead further produced a very elegant geometrical conception of the Postnikov invariants in a paper [19] published in 1953. In this paper he pointed out that if K is a CW-complex we may form a nest of complexes... $K^{(r)} \subseteq K^{(r-1)} \subseteq \ldots$ such that $K^{(r)}$ is obtained from K by killing the homotopy groups of K above the r^{th}. The first obstruction to retracting $K^{(r-1)}$ onto $K^{(r)}$ is an element of $H^{r+1}(K^{(r-1)}; \pi_r(K))$ which is the appropriate Postnikov invariant. Unfortunately the title of this paper, *On the G-dual of a semi-exact couple*, prevented it from enjoying the currency it merited.

I would like to mention now one other original contribution that Whitehead made to algebraic homotopy theory, this time in collaboration with E. H. Spanier; I refer of course to S-theory, which describes a duality in what may be called an approximation to homotopy theory. Given spaces X, Y (with base points), an S-map $X \to Y$ is a map $f\colon S^m X \to S^m Y$ for some m (where S^m is the m-fold suspension) and $f\colon S^m X \to S^m Y$, $g\colon S^n X \to S^n Y$ are S-equivalent if $S^{r-m} f \simeq S^{r-n} g$ for some r. The S-equivalence classes of S-maps form an abelian group $\{X, Y\}$; if $X = S^k$, this is the " stable k^{th} homotopy group " of Y and if $Y = S^k$ and X is compact, dim $X \leqq 2k - 2$, then $\{X, Y\}$ is the k^{th} cohomotopy group of X.

Now let X be a subpolyhedron of S^n and let $D_n X$ be a subpolyhedron of $S^n - X$ which is a S-deformation retract of $S^n - X$. Then $D_n X$ is called an n-dual of X; one observes that then $D_n X$ is an $(n+1)$-dual of SX. If also $D_n Y$ is an n-dual of Y there is defined a canonical isomorphism

$$D_n\colon \{X, Y\} \cong \{D_n Y, D_n X\}\,.$$

Certainly X is an n-dual of $D_n X$ and the duality is expressed by the relation

$$D_n D_n \alpha = \alpha, \quad \alpha \in \{X, Y\}\,.$$

The map D_n has many highly desirable properties of a functorial nature; in addition it is related to Alexander duality by the commutative diagram

$$\begin{array}{ccc} H_p(X) & \xrightarrow{\alpha_*} & H_p(Y) \\ \updownarrow & & \updownarrow \\ H^{n-p-1}(D_n X) & \xrightarrow{(D_n\alpha)^*} & H^{n-p-1}(D_n Y) \end{array}$$

To remove the restriction of embeddability, one may define a relation of weak n-duality between finite CW-complexes X and X^* by asking that X and X^* should be S-equivalent to n-dual polyhedra X_0 and $D_n X_0$ respectively. Recently Spanier has further simplified the concept of n-duality by merely asking for a map

$$u\colon X^* \# X \to S^{n-1}$$

such that the induced slant product yields isomorphisms

$$H_q(X^*) \cong H^{n-q-1}(X)$$

for all q; reasonably, Spanier now calls this $(n-1)$-duality so we again meet the standard correction term ± 1 of homotopy theory. The Spanier-Whitehead theory is rich and satisfying; the duality between homotopy and cohomotopy is elucidated, attaching " dual " cones to " dual " bases by " dual " maps produces " dual " spaces, there is a thoroughgoing relativization, and the whole story serves to illuminate the essential features of the suspension range. In concluding this section, I would like to mention a most elegant application made recently by Atiyah; if τ is the tangent bundle to a closed differentiable manifold X and α is a real vector bundle over X then the Thom complexes X^α and $X^{-\alpha-\tau}$ are S-dual.

There are, of course, numerous further examples that could be quoted of Whitehead's work. His discovery of the Whitehead product [20], essentially the only non-trivial primary homotopy operation, deserves mention. His study of the Freudenthal suspension theorems revealed his usual meticulous attention to details, together with his ability to grasp the essentials of another's work; and I for one owe to his *Note on Suspension* [21] my understanding of the original Freudenthal arguments. In fact Whitehead very substantially generalized the scope of Freudenthal's results, and showed in effect how to use knowledge of the stable homotopy groups of spheres to compute stable homotopy groups of polyhedra. But I would rather devote the remainder of the time available to me to speak a little of Whitehead the man.

Whitehead has immense achievements in mathematics to his credit; but it was impossible to tell from meeting him and talking to him that one was in the presence of great eminence. His friendliness and informality are well-known to you all and his success in building up a research school in topology in Oxford owes a great deal to the qualities of character which Whitehead brought to this task. He was far and away the most dynamic personality among Oxford mathematicians in the ten years following the war, and I must be forgiven the apparent disrespect to his colleagues if I say that it was natural and inevitable that almost all the bright young students should on graduating choose to do their research under his guidance. Whitehead was an inspiring but unorthodox research supervisor; for his method —at any rate as it revealed itself to me—was simply to treat his student as an intimate colleague and thus to involve him completely in the mathematical interests which were at that particular time preoccupying him. The stimulus of such close and utterly informal contact cannot be overemphasized; the research student might be overawed and overwhelmed, but there were no dull moments.

Whitehead loved mathematics and he was intensely serious about it. I recall an occasion when he sought to dissuade a young mathematician from abandoning an academic career. The young man's grounds were mainly that he would never be a first class mathematician but believed that he could be first-class in the more restricted field he intended to enter. To this argument Whitehead's reply was characteristic: " far better to be second-rate in a first-rate line that first-rate in a second-rate line ". To Whitehead mathematics was one among a small number of human activities which were intrinsically worthwhile; and from such activities one should if possible choose one's career. He hated compromise on this question—he strongly disapproved of those who chose their jobs within the mathematical world on the basis of potential size of income, arguing that, if you want money, you should take a job appropriate to that want and not try to keep a foot in both camps. I think I am being fair in stating his position thus. I certainly do not imply that Whitehead was opposed to mathematicians receiving handsome remuneration, but he was strongly opposed to the supplementation of one's income by non-mathematical activities taking up a substantial proportion of potential working time. He himself was happy to devote himself unreservedly to mathematics and would, for days on end, work a 15-hour day interrupted briefly for meals. Such a regimen was intimidating to a young research student seeking to discover the secret of success in mathematical creation, but it was inspiring too. Whitehead always made it clear to his students that mathematics was impossible without real hard work. It is not sufficient to have bright ideas; these must be worked out and elaborated in detail and it was impossible to do this without getting your hands dirty. He attributed, in many cases, the failure of a mathematician to fulfil the promise of his youth precisely to the fact that, having got the bright ideas, the man in question was reluctant to involve himself in the labour of pursuing them. A further precept that he passed on to his students—and, to be sure, to others, too—was the necessity of understanding any result quoted. In the first instance Whitehead was doubtless influenced by his own experience in using results subsequently shown to be wrong or inaccurate (you will recall the false determination of $\pi_5(S^3)$, and also Whitehead's letter in the *Annals* in 1953, correcting a whole host of incorrect signs); but there was both a positive and a negative side to this precept, for Whitehead argued that, if you understood the proof of a result you understood that much better its significance, too. He himself declared the reading of other's papers to be the hardest part of the job, but it was one he never shirked; and his use and development of the work of others testifies unmistakably to the success of this part of his method. (It might perhaps be added parenthetically that Whitehead might have found the reading of others' works easier than we do; for we have to read Whitehead's works !) While discussing his fundamental seriousness of purpose, I think I may be forgiven for mentioning his often reiterated attitude towards conferences and colloquia. He loved to attend them and revelled in the company of colleagues and friends. But he always maintained that they were no alternative to hard, painstaking thought in the privacy of one's own study—although there were those who seemed to regard them as such an alternative. I remember how his conscience seemed to be worrying him in Mexico in 1956. Probably many of your recollections of his participation in that symposium tend to be in terms of animated conversation, tequila and fronton; but I, who shared his bedroom, can tell you that many hours which I would have expected to have spent asleep were in fact devoted to an attempt to assure Henry that he wasn't defrauding the taxpayers by his attendance, and that good mathematics might well emerge even from so delightful a social experience as the Mexico Symposium.

I have come nearly to the end of my remarks. A man of immense intellectual and physical vigour, of great charm and friendliness, honest, sincere and conscientious, has died. No longer will a home in Charlbury Road or a farm in Noke be the first place that any one of us, in coming to Oxford, will think of visiting. England and the world have lost a great man and many of us here have lost a dear friend. But it is fitting that this tribute should be paid to Whitehead in the midst of a colloquium in which we are both looking forward at new horizons in mathematics and back at the achievements of recent years. For certainly Whitehead would never have us abstain from doing either; and whichever way we look we find inspiration in his life and work.

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