2. Realizing a graded algebra as a cohomology ALGEBRA.

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$\{A^q\}$, $q = 0, 1, 2, \ldots$, of R-modules. Thus, an element of A is an element of some A^q, and q is called its degree. Elements may be added only if they have the same degree. In addition, homomorphisms $A^p \otimes A^q \to A^{p+q}$ are given for all $p, q \geqq 0$. These define a bilinear product xy for all $x, y \in A$. The product is required to be associative.

An ordinary algebra C is converted into a graded algebra A by setting $A^0 = C$ and $A^q = 0$ for $q > 0$. In this way, a graded algebra is a *generalization* of the notion of an algebra. Thus we are free to generalize the properties of algebras to graded algebras in any convenient manner which conforms in degree 0.

In particular, a graded algebra A is called commutative if $xy = (-1)^{pq}\, yx$ for all $x \in A^p$ and $y \in A^q$. Thus, what was once called the *anti*-commutative law is now called the commutative law. And the cohomology algebra of a space is an associative, commutative, graded algebra.

A unit of a graded algebra A is an element $1 \in A^0$ such that $1x = x = x1$ for all $x \in A$. An augmentation ε of A is a homomorphism $\varepsilon\colon A \to R$ of graded algebras with unit. Thus $\varepsilon(A^q) = 0$ for $q > 0$, and $\varepsilon(1) = 1$. In case ε gives an isomorphism in degree 0, $A^0 \approx R$, then A is called *connected.*

If P is a space consisting of a single point, it is clear that $H^*(P; R) = R$ as a graded algebra. For any space X, the mapping $\eta\colon X \to P$ induces a monomorphism $\eta^*\colon R \to H^*(X; R)$ and $\eta^*(1)$ is the unique unit of $H^*(X; R)$. Finally, any mapping $P \to X$ induces an augmentation of $H^*(X; R)$. Clearly X is arcwise connected if and only if $H^*(X; R)$ is connected; and then the augmentation is unique.

2. Realizing a graded algebra as a cohomology algebra.

Let Z denote the ring of integers. It is well known that, if B is a graded Z-module such that $B^0 = Z$, B^1 is free, and B^n is finitely generated for each n, then there is a space X which realizes B in that $H^*(X; Z) \approx B$. One solves this problem, for a single dimension n, by a cluster C_n of n-spheres and $(n+1)$-cells; and then the general case is solved by a union of the C_n's with a common point. If Z is replaced by another of the simpler ground rings, there are no serious difficulties in solving the realization problem.

Suppose however that B is a graded, commutative, associative and connected algebra over R. A *realization* of B is a space X such that $H^*(X; R) \approx B$ as graded algebras. The problem of deciding when a given B is realizable has not been solved, and is very difficult. To make the problem precise, we shall use singular cohomology groups, and require X to be a CW-complex. A natural attack on the problem is to consider first the case of certain simple B's, and then pass to more complicated ones.

Let $F(R, n)^\infty$ denote the graded, free, commutative, associative, and connected algebra over R on one generator x of dimension n; and let $F(R, n)^h$ be the quotient algebra obtained by setting $x^h = 0$ (h = height of x). Thus, if n is even, $F(R, n)^\infty$ is the " polynomial " algebra of x, i.e. the monomials $1 = x^0, x^1, \ldots, x^k, \ldots$ form a module basis; and it is a free R-module. If n is odd, the commutative law demands that $2x^2 = 0$, so $2x^k = 0$ for all $k \geqq 2$. Thus the kn dimensional part is isomorphic to $R/2R$ for $k \geqq 2$. Setting $x^h = 0$ replaces all component groups in dimensions $\geqq hn$ by zero.

We will discuss the problem of realizing $F(R, n)^h$ in the special cases where R is the ring Z of integers, or the field Z_p of integers reduced modulo a prime p. First, we will list three trivial cases.

The space consisting of a single point realizes $F(R, n)^1$ for all R and n.

The n-sphere S^n realizes $F(R, n)^2$ for all R and n.

If n is odd, S^n also realizes $F(R, n)^h$ for all $2 \leqq h \leqq \infty$ providing $R/2R = 0$, because the relation $2x^2 = 0$ must then imply that $x^2 = 0$; so h is effectively 2. The condition $R/2R = 0$ holds for $R = Z_p$, $p > 2$.

The projective spaces over the real numbers, complex numbers and quaternions provide realizations of quite a few of the F's. Consider first the real projective n-space P^n. Taking $R = Z_2$, $H_q(P^n) \approx Z_2$ for $0 \leqq q \leqq n$, and the non-trivial element is represented by any subspace $P^q \subset P^n$. Now any P^q

can be made the intersection of $n-q$ projective $(n-1)$-subspaces. The duality between intersections of cycles and products of cocycles shows that $H^*(P^n; Z_2) \approx F(Z_2, 1)^{n+1}$. Let P^∞ be the union of a sequence $P^0 \subset P^1 \subset \ldots \subset P^n \subset \ldots$. Then P^∞ realizes $F(Z_2, 1)^\infty$.

The complex projective n-space CP^n has real dimension $2n$. It has no torsion, its odd dimensional Betti numbers are zero, and its Betti number is 1 in each even dimension $\leqq 2n$. A generator of $H_{2q}(CP^n; R)$ is provided by any projective subspace $CP^q \subset CP^n$. Again, the duality between intersections and products shows that $H^*(CP^n; R) \approx F(R, 2)^{n+1}$ for any R. Forming CP^∞, as above, realizes $F(R, 2)^\infty$.

The quaternionic projective n-space QP^n has real dimension $4n$, no torsion, and non-zero Betti numbers equal to 1 in dimensions $4q \leqq 4n$. A similar argument shows that QP^n realizes $F(R, 4)^{n+1}$ for each R and each $n \leqq \infty$.

The Cayley numbers (on 8 units) is non-associative. As a result the usual notion of the equivalence of two sets of homogeneous coordinates fails to be transitive; hence there is no Cayley projective n-space. An exception is $n = 2$, because any two Cayley numbers generate an associative subalgebra. Using this, Hopf [11] constructed a Cayley projective plane M of real dimension 16. It has no torsion, and its non-zero Betti numbers are equal to 1 in dimensions 0, 8 and 16. An appropriate argument shows that M realizes $F(R, 8)^3$ for any R.

The preceding results are very encouraging, a great many of the F's are realized by spaces which are not too complicated. One might be led by these to expect that any F can be realized. A bit of ingenuity in putting spaces together should do the trick. Once the case of one generator is thus solved, the special case of many generators given by a tensor product of F's for various n's and h's, can be solved by cartesian products of the separate realizations. Thus it begins to appear likely that any graded, commutative and associative algebra can be realized.

The historical fact is that topologists were lulled to sleep by the above considerations. Their preconception of the nature of the cohomology algebra appeared to be justified. However they were awakened abruptly in 1952 by the result of Adem [2] which states: If n is not a power of 2, and $3 \leqq h \leqq \infty$, then $F(Z_2, n)^h$ cannot be realized.

Subsequent revelations showed that the situation is even worse: the preceding examples of realizations of F's are nearly all that exist. The method for proving this uses the fact that the cyclic reduced pth powers, which operate in the algebra $H^*(X; Z_p)$, satisfy certain relations. In the next three sections we will discuss these operations, and their implications for the realization problem.

3. Construction of the squaring operations.

Before presenting the algebra of the reduced power operations, it may be worthwhile to give a recently improved form of the definition of the operations themselves. For simplicity we restrict ourselves to the case of the prime 2.

Let π be a cyclic group of order 2 with generator T ($T^2 = 1$). Let W be an acyclic complex on which π acts freely. Algebraically, W is a free resolution of Z over π. Geometrically, W can be taken to be the union of an infinite sequence of spheres $S^0 \subset S^1 \subset \ldots \subset S^n \subset \ldots$ where each is the equator of its successor, and T is the antipodal transformation. Identifying equivalent points under π gives the infinite dimensional real projective space P with W as its 2-fold covering. Recall that $H^*(P; Z_2)$ is the polynomial ring $F(Z_2, 1)^\infty$ on the one dimensional generator U.

Let K be any space, form the cartesian product $W \times K \times K$, and let π act in this space by $T(w, x, y) = (Tw, y, x)$. Then T has no fixed points. Identifying equivalent points gives a space, denoted by $W \times_\pi K^2$, which is covered twice by $W \times K^2$. Imbed $W \times K$ in $W \times K^2$ by $(w, x) \to (w, x, x)$. Then π transforms $W \times K$ into itself with $T(w, x) = (Tw, x)$. It follows that $W \times K$ covers a subspace $P \times K$ imbedded in $W \times_\pi K^2$. This gives the diagram

$$(3.1) \qquad P \times K \xrightarrow{i} W \times_\pi K^2 \xleftarrow{h} W \times K^2 \xrightarrow{g} K^2$$

where i is the inclusion, h is the covering, and g is the obvious projection.