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is decomposable in terms of secondary operations for each $i \geqq 1$. Using this result, 5.2 can be improved as follows:

5.8. *If* n *is not a divisor of* $p - 1$, *and* $p < h \leqq \infty$, *then* $F(Z_p, 2n)^h$ *cannot be realized.*

This leaves a good many unsettled cases. For example can $F(Z_p, 2(p-1))^3$ be realized for some $p > 5$? Can $F(Z_5, 8)^4$ be realized ? The cohomology of such a space would necessarily have torsion involving the prime 3. Likewise unsettled are the cases of $F(Z, 2n+1)^h$ where $n > 1$, $h > 2$ and $n = 1$, $h > 3$. In view of the preceding results, it seems unlikely that any of these can be realized.

For a rough summary, let us exclude the trivial cases $h = 1, 2$. Then the only n's for which $F(R, n)^h$ is known to be realizable are included among the integers 1, 2, 4 and 8. If $R = Z, Z_2$, or Z_3 it is not realizable for any other n. If $R = Z_p$, it is not realizable for $h > p$ and $n > 2(p-1)$. In short, $F(R, n)^h$ is not realizable except in rare cases involving small values of n or h.

These negative conclusions have interesting implications in algebra. The successful realizations were obtained by using projective spaces over the real numbers, complex numbers, quaternions, and Cayley numbers. If there is a real division algebra on n units, we can use it to realize $F(Z_2, n)^3$; hence our non-existence results imply that $n = 1, 2, 4$ or 8. Again, since $F(Z_3, 8)^4$ is not realizable, it follows that there is no real, associative division algebra on 8 units.

6. Hopf algebras.

Historically, we started with the preconception that the cohomology of a space is nothing more than a graded algebra, and we asked if certain simple graded algebras could be realized. On the whole we found that the answer was negative; and this was shown by using the fact that the algebra $\mathscr{A}_p$ of reduced powers operates in $H^*(X; Z_p)$. Our preconception was misleading, the cohomology algebra of a space is something more than a graded algebra. Just how much more is not yet clear.

However a certain part of this additional structure can be clarified; and we shall do so in this and subsequent sections.

Let us recall the concept of a Hopf algebra A. In the first place A is a graded, associative algebra over the ground ring R with a unit and an augmentation $\varepsilon: A \to R$. The unit is regarded as a homomorphism of algebras $\eta: R \to A$ defined by $\eta(1_R) = 1_A$. Define $A \otimes A$ to be the graded module whose component of degree r is given by

$$(A \otimes A)_r = \sum_{q=0}^{r} A_q \otimes A_{r-q} .$$

The multiplication mappings $A_p \otimes A_q \to A_{p+q}$ are the components of a mapping $\varphi: A \otimes A \to A$ of graded R-modules. Define an algebra structure in $A \otimes A$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{qr}(aa') \otimes (bb')$$

where $q = \deg b$, and $r = \deg a'$. The final element of structure is a " diagonal mapping "

$$\Psi: A \to A \otimes A$$

which is required to be a homomorphism of algebras with unit, and to satisfy the conditions

$$(\varepsilon \otimes 1)\Psi a = 1 \otimes a , \quad (1 \otimes \varepsilon)\Psi a = a \otimes 1$$

as mappings $A \to R \otimes A$, and $A \to A \otimes R$.

Furthermore, Ψ is usually required to be *associative*, i.e. the mappings $(1 \otimes \Psi)\Psi$ and $(\Psi \otimes 1)\Psi$ of A into $A \otimes A \otimes A$ coincide. Sometimes Ψ is required to be *commutative*, i.e. $T\Psi = \Psi$ where $T: A \otimes A \to A \otimes A$ is defined by $T(a \otimes b) = (-1)^{pq} b \otimes a$ where $p = \deg a$, $q = \deg b$. In most applications, φ or Ψ is commutative, but rarely both.

The Hopf algebra structure thereby consists of the mappings

$$A \xrightarrow{\Psi} A \otimes A \xrightarrow{\varphi} A , \quad R \xrightarrow{\eta} A \xrightarrow{\varepsilon} R .$$

The asymmetry in Ψ, φ and η, ε gives rise to a duality. The graded module A together with a mapping $\Psi: A \to A \otimes A$ is called a *coalgebra* and $\varepsilon: A \to R$ is called a unit for the coalgebra. The requirement that Ψ be a homomorphism of algebras is equivalent to demanding that φ be a homomorphism of coalgebras. This compatibility of φ, Ψ is expressed in a neutral fashion by requiring that the following diagram be commutative:

$$\begin{tikzcd}
A\otimes A \arrow[rr, "\varphi"] \arrow[d, "\Psi\otimes\Psi"] & & A \arrow[dd, "\Psi"] \\
A\otimes A\otimes A\otimes A \arrow[dr, "1\otimes T\otimes 1"] & & \\
& A\otimes A\otimes A\otimes A \arrow[r, "\varphi\otimes\varphi"] & A\otimes A
\end{tikzcd}$$

The concept of Hopf algebra arose first in Hopf's study [12] of the homology of a group manifold G. The diagonal mapping and the multiplication mapping

$$G \xrightarrow{\Psi} G \times G \xrightarrow{\varphi} G$$

induce homomorphisms of homology over a field of coefficients

$$H_*(G) \xrightarrow{\Psi_*} H_*(G) \otimes H_*(G) \xrightarrow{\varphi_*} H_*(G)$$

and the group homomorphisms $1 \rightarrow G \rightarrow 1$ induce the unit and augmentation. In this case Ψ_* is commutative. If, instead, we pass to cohomology, then φ^* becomes the diagonal mapping, and the multiplication Ψ^* is commutative.

Because of this application to Lie groups, Hopf algebras have been studied extensively. One of the best results, due to Borel [5], assumes that R is a perfect field of characteristic p and A has a commutative multiplication $A_0 \approx R$ and A_q is of finite rank for each q. The conclusion is that, as an algebra, A is a tensor product of subalgebras B^1, B^2, ... each on a single generator b_1, b_2, ... If $p>2$ and deg b_i is odd, B^i is an exterior algebra ($b_i^2 = 0$); and if $p = 2$, or if $p > 2$ and deg b_i is even, B^i is either the polynomial ring on b_i, or the polynomial ring truncated by the relation $b_i^h = 0$ where h is a power of p.

It was Milnor [14] who observed that the reduced power algebra $\mathscr{A}_p$ is a Hopf algebra with the diagonal mapping defined by

$$\Psi \mathscr{P}^k = \Sigma_{i=0}^k \mathscr{P}^i \otimes \mathscr{P}^{k-i}\,, \qquad \Psi\beta = \beta \otimes 1 + 1 \otimes \beta\,.$$

That Ψ is a homomorphism of algebras follows from 4.11. In this case Ψ is commutative; so the dual Hopf algebra $\mathscr{A}_p^*$ has a commutative multiplication. Milnor found an explicit and simple analysis of the structure of $\mathscr{A}_p^*$ as a tensor product of an exterior algebra and a polynomial algebra. Using an equally explicit form for the diagonal of $\mathscr{A}_p^*$, he was able to obtain results on the structure of $\mathscr{A}_p$ as an algebra. In particular, it is nilpotent.

It is to be emphasized that Hopf algebras have arisen in algebraic topology in these two very natural but quite different ways. This suggests that the concept is even more fundamental than had been thought. The next sections are devoted to developing the theme that Hopf algebras are basic because there are strong, purely algebraic reasons for introducing them.

7. Modules over Hopf algebras.

As a preliminary, let us review certain facts about the category $C(R)$ of graded modules over the ground ring R. The two functors $X \otimes Y$ and $\mathrm{Hom}(X, Y)$, where $\otimes$ and Hom are taken over R, have values in $C(R)$ when X, Y are in $C(R)$. The gradings of $X \otimes Y$ and $\mathrm{Hom}(X, Y)$ are defined by

$$(X \otimes Y)_r = \sum_{p+q=r} X_p \otimes Y_q$$

$$\mathrm{Hom}(X, Y)_r = \prod_{q-p=r} \mathrm{Hom}(X_p, Y_q) .$$

The index of the gradings ranges over all integers.

Furthermore, there are natural equivalences

$$(7.1) \qquad R \otimes X \approx X \approx X \otimes R\,, \quad \mathrm{Hom}(R, X) \approx X$$

obtained by identifying $r \otimes x = rx = x \otimes r$, and $f = f(1)$ for $f \in \mathrm{Hom}(R, X)$. The commutative law

$$(7.2) \qquad T \colon X \otimes Y \approx Y \otimes X$$

is a natural equivalence defined by $T(x \otimes y) = (-1)^{pq}\, y \otimes x$ where $x \in X_p$ and $y \in Y_q$. The associative law