10. Reformulation of the problem.

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Form now the quotient of $T(M)$ by the ideal N generated by elements

$$(9.2) \qquad x \otimes y - (-1)^{pq} y \otimes x \text{ where } x \in M_p , \quad y \in M_q .$$

The quotient, denoted by $U(M)$, is called the *free, commutative and associative algebra generated by* M. If we assume that the diagonal mapping Ψ of A is commutative, then it is readily verified that N is an A-submodule of $T(M)$. Hence $U(M)$ becomes an algebra over the Hopf algebra A.

As is well known, the algebra $T(M)$ is *universal* in the sense that any R-mapping of M into an algebra X extends to a unique mapping of algebras $T(M) \to X$. Furthermore, if X is an algebra over A, and $M \to X$ is an A-mapping, so also is $T(M) \to X$. A similar statement holds for $U(M)$ in case X is commutative.

Additional algebras over A can be constructed by taking a submodule of $T(M)$ or $U(M)$ forming the A-ideal it generates, and passing to the quotient algebra. It is easily seen that any A-algebra can be obtained as such a quotient.

In the special case where A is the algebra $\mathscr{A}_p$ of reduced powers, only certain M's are admissible, namely, those which satisfy the dimensionality restriction 4.9: $\mathscr{P}^i x = 0$ whenever $2i > \dim x$. Moreover, in forming $U(M)$, we must increase the ideal N so as to include all elements of the form

$$(9.3) \quad \mathscr{P}^k x - (x \otimes x \otimes \ldots \otimes x) \, (p \text{ factors}) , \quad x \in M_{2k} .$$

This insures that the relation 4.8, namely, $\mathscr{P}^k y = y^p$ is valid for $y \in U(M)_{2k}$. (It is a pleasant exercise in the use of the Adem-Cartan relations to show that N is an $\mathscr{A}_p$-module.) With these modifications, the resulting $U(M)$ is meaningful for algebraic topology.

10. Reformulation of the problem.

We are now in a position to formulate a problem similar to the one posed in section 2, but having a better chance of a positive solution. Recall that the algebra $F(R, q)^\infty$ of section 2 is small in that it has a single generator but is otherwise as big as possible subject to being commutative and associative. We found that, for many q's, it is not an $\mathcal{A}_p$-algebra, and hence cannot be realized. In analogy, we shall construct $U(Z_p, q)$ the free, commutative, associative $\mathcal{A}_p$-algebra on one generator of dimension q.

In $\mathcal{A}_p$, let $N(q)$ be the left ideal spanned by monomials in β and the $\mathcal{P}^i$ each of which has a factored form $Q' \beta^{\varepsilon} \mathcal{P}^k Q$ where $2k + \varepsilon > q + \deg Q$ and $\varepsilon = 0$ or 1. By 4.9, any such a monomial gives zero when applied to a q-dimensional class. Set $M(q) = \mathcal{A}_p/N(q)$ and define dimension by adding q to the degree in $\mathcal{A}_p$. Then $M(q)$ is an $\mathcal{A}_p$-module, the admissibility condition 4.9 holds, it has a single $\mathcal{A}_p$-basis element of dimension q, and it is the largest admissible $\mathcal{A}_p$-module on one element of dimension q. Finally, set $U(Z_p, q) = U(M(q))$ as defined in section 9.

If now we ask whether $U(Z_p, q)$ is realizable, the answer is Yes! It has been proved by Cartan [7] that $U(Z_p, q)$ is isomorphic as an $\mathcal{A}_p$-algebra to the cohomology algebra of the Eilenberg-MacLane complex $K(Z_p, q)$.

Having succeeded in realizing the free $\mathcal{A}_p$-algebra on one generator, it is natural to ask if quotients of this algebra can be realized. For example, choose a $y \in U(Z_p, q)$ and let W be the quotient by the minimal $\mathcal{A}_p$-ideal containing y. As an approach to this question, let D be the canonical bundle over $K(Z_p, q)$ with y as its k-invariant. Precisely, the element $y \in H^r(K(Z_p, q), Z_p)$ determines a mapping $f \colon K(Z_p, q) \to K(Z_p, r)$ such that y is the image of the fundamental class of $K(Z_p, r)$. Let E be the acyclic fibre space over $K(Z_p, r)$ with fibre $K(Z_p, r - 1)$. Then D is defined to be the fibre space over $K(Z_p, q)$ induced by E and f.

Unfortunately the complete structure of $H^*(D; Z_p)$ is not known. It is obvious that the projection $g \colon X \to K(Z_p, q)$ satisfies $g^* y = 0$. Therefore the kernel of g^* contains the $\mathcal{A}_p$-ideal generated by y. It is a reasonable conjecture that they coincide, and that the $\mathcal{A}_p$-algebra W on one generator and one relation is contained in $H^*(D; Z_p)$. It is definitely known that W is not all of $H^*(D; Z_p)$. To see this, it is only necessary to recall that the elements of $H^*(K(Z_p, q); Z_p)$ can be interpreted as primary cohomology operations, and the elements of $H^*(D; Z_p)$ as secondary operations defined on cohomology classes annihilated by y (see [1]). Numerous non-trivial secondary operations have been found.

Thus to realize W as the cohomology algebra of a space, we must modify D so as to eliminate the unwanted elements of $H^*(D; Z_p)$. But before trying this, we should reexamine our objective. It was to construct a space whose cohomology has a single generator and is maximal subject to a single relation. In one sense D already satisfies our requirement. If we admit *secondary* cohomology operations as well as the primary operations $\mathscr{A}_p$, then the g^*-image of the generator of $H^*(K(Z_p, q); Z_p)$ does in fact generate $H^*(D; Z_p)$, and the latter is free in the sense that there are no accidental relations. This is a restatement of the identification of elements of $H^*(X; Z_p)$ with secondary operations associated with y.

Thus, in attempting to realize W, we have tacitly assumed that we know what is meant by " one generator subject to one relation ". Our prejudices have again interposed themselves. The correct procedure is to analyse fully the structure of $H^*(D; Z_p)$, and then we may know how to define the concept of one generator subject to one relation.

Eventually we will want to know how to describe algebraically the cohomology algebra on k generators subject to r_1 primary relations, r_2 secondary relations, etc. We know already how to realize this algebra using Eilenberg-MacLane complexes and the fibre space constructions of Postnikov [16]. But we are a long way from being able to describe the algebra in direct algebraic terms.

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