APPROXIMATING SUMS AND THE TEACHING OF THE RIEMANN INTEGRAL

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The limit process appearing in the definition of the Riemann integral, is a difficult one for a student to comprehend, even if he has already grasped the notion of " the limit of a single-valued function ".

The following method of presentation is designed to provide the student with a conceptual bridge between these two notions of limit.

The approach is especially suitable to the classroom because appeal can be made, at any step in the method, to geometrical relationships.

The results of the classical theory of Riemann integration are assumed.

First, a few standard definitions are needed:

A *partition* of $[a, b]$ [1]) is a finite set of points $x_0, x_1, ..., x_n$ such that $a = x_0 < x_1 < ... < x_n = b$. Such a partition will be denoted by $(a, x_1, ..., b)$.

The *norm* of $(a, x_1, ..., b)$ is the maximum of the differences, $x_j - x_{j-1}$ for $j = 1, ..., n$.

A *mesh* of $(a, x_1, ..., b)$ is any of the closed subintervals $[x_{j-1}, x_j]$ for $j = 1, ..., n$.

If f is a continuous real-valued function defined at every point of $[a, b]$, and $(a, x_1, ..., b)$ is a partition of $[a, b]$, then $\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$, where $x_{j-1} \leqq c_j \leqq x_j$ for $j = 1, ..., n$, is called a *Riemann sum* for $(a, x_1, ..., b)$.

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1) $[a, b]$ denotes the closed interval from a to b.

If, in the preceding definition, $f(c_j)$ is the maximum value of f on $[x_{j-1}, x_j]$ for $j = 1, ..., n$, then the Riemann sum is called the *upper Riemann sum* for $(a, x_1, ..., b)$. If, on the other hand, $f(c_j)$ is the minimum value of f on $[x_{j-1}, x_j]$ for $j = 1, ..., n$, then the Riemann sum is called the *lower Riemann sum* for $(a, x_1, ..., b)$.

In the following approach detailed study is made of the behavior of the Riemann sums for some simple function. The method will be applied to the function $y = x^2$ on $[0,1]$.

Let $y = x^2$ on $[0,1]$. Let h_0 denote any real number such that $0 < h_0 \leqq 1$. Let $K(h_0)$ denote the set of all partitions each of which has a norm less than, or equal to, h_0. Let $S(h_0)$ denote the set of all numbers that are Riemann sums for at least one partition belonging to the set $K(h_0)$.

THEOREM I. If $y = x^2$ on $[0,1]$ and h_0 is such that $0 < h_0 \leqq 1$, then the least upper bound of $S(h_0)$ is attained by the upper Riemann sum for the partition $P^*(h_0) = (0,1 - nh_0, 1 - (n - 1)h_0, ..., 1 - h_0, 1)$ where $n = [1/h_0]$, that is, n is the greatest integer less than, or equal to, $1/h_0$. For example, if $1/4 < h_0 \leqq 1/3$, then $[1/h_0] = 3$, and the least upper bound of $S(h_0)$ occurs for the partition $P^*(h_0) = 0,1 - 3h_0, 1 - 2h_0, 1 - h_0, 1)$.

Proof. In the search for a least upper bound for $S(h_0)$, we need merely consider upper Riemann sums. Let $P(h_0) = (0, x_1, ..., 1)$ denote any other partition of norm h_0 [1]). It will be proved that the upper Riemann sum for $P(h_0)$ is less than, or equal to, the upper Riemann sum for $P^*(h_0)$. The meshes of $P(h_0)$ can be segregated into disjoint classes C_1 and C_2 as follows: C_1 consists of those meshes of $P(h_0)$ which are subsets (proper or improper) of a mesh of $P^*(h_0)$, and C_2 consists of the remaining meshes of $P(h_0)$, that is, those meshes of $P(h_0)$ which contain a point [2]) of $P^*(h_0)$ in their interior. Now, the maximum of $y = x^2$ over a mesh belonging to C_1 is less than, or equal to, the maximum of $y = x^2$ over the mesh of

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1) Note that $P^*(h_0)$ also has norm h_0. In Figures I, II, and III, the solid lines denote rectangles associated with the partition $P^*(h_0)$ and the dotted lines denote rectangles related to $P(h_0)$.

2) No mesh of $P(h_0)$ can contain more than one point of $P^*(h_0)$ in its interior because the norm of $P(h_0)$ is h_0.

$P^*(h_0)$ of which it is a subset. Thus, the contribution to the upper Riemann sum of $P(h_0)$ by a rectangle, with a mesh from C_1 as base, is consistent with the offered conclusion of the theorem.

Let us now turn to the consideration of the contribution to the upper Riemann sum of $P(h_0)$ by a rectangle erected on a mesh that belongs to C_2. Let $[x_{j-1}, x_j]$ denote a mesh of C_2 that contains $1 - kh_0$, a point of $P^*(h_0)$, in its interior. Note that $0 < x_j - x_{j-1} \leqq h_0$ and $x_{j-1} < 1 - kh_0 < x_j$. Figure I will aid in the analysis.

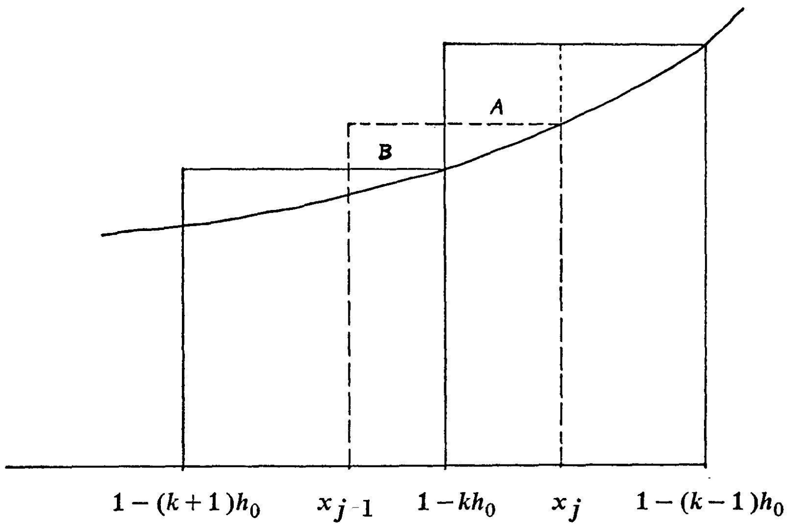

Fig. I.

It will be proved that the area of the rectangle denoted by B is less than the area of the rectangle denoted by A. Thus, the upper Riemann sum for $P(h_0)$ " loses " more (A) than it " gains " (B) [1].

The area of rectangle B is

$$[(1 - kh_0) - x_{j-1}]\ [x_j^2 - (1 - kh_0)^2]$$

and the area of rectangle A is

$$[x_j - (1 - kh_0)]\ [\{1 - (k - 1)h_0\}^2 - x_j^2]\,.$$

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1) An analogous result does not hold for an arbitrary monotone continuous function. For example, consider the function obtained from $y = x^2$, by substituting a line segment through the points (x_j, x_j^2) and $(1 - (k - l)h_0, x_j^2 + e)$, where $e > 0$, for the portion of $y = x^2$ defined over the interval $[x_j, 1 - (k - l)h_0]$. Since e can be made arbitrarily small, we can obviously construct a monotone continuous function for which the area of A is less than the area of B.

Now, since $x_j - (1 - kh_0) > 0$, it suffices to prove that

$$[(1-kh_0) - x_{j-1}]\,[x_j + (1-kh_0)] < \{1 - (k-1)h_0\}^2 - x_j^2 .$$

Observe that

$$[(1-kh_0) - x_{j-1}]\,[x_j + (1-kh_0)] \leqq [(1-kh_0) - x_{j-1}]\,[(x_{j-1} + h_0) + (1-kh_0)]$$

and that

$$\{1 - (k-1)h_0\}^2 - (x_{j-1} + h_0)^2 \leqq \{1 - (k-1)h_0\}^2 - x_j^2 .$$

Therefore, it suffices to prove that

$$[(1-kh_0) - x_{j-1}]\,[(x_{j-1} + h_0) + (1-kh_0)] < \{1 - (k-1)h_0\}^2 - (x_{j-1} + h_0)^2 .$$

By simple manipulation, one obtains $x_{j-1} < 1 - kh_0$. But this inequality is given. The fact that the area of rectangle A is greater than that of rectangle B follows from a reversal of the previous steps. Combining our results, one notes that the upper Riemann sum for $P(h_0)$ is less than, or equal [1]) to, the upper Riemann sum for $P^*(h_0)$ and thus, Theorem I is proved.

THEOREM II. Same hypotheses as Theorem I. Then the least upper bound $\bar{S}(h_0)$ of $S(h_0)$ is given by the formula:

$$\bar{S}(h_0) = 1 - 2nh_0 + h_0^2 n(2n+1) - \frac{h_0^3 n}{6}(4n-1)(n+1) ,$$

where $n = [1/h_0]$.

THEOREM III. If $y = x^2$ on $[0,1]$ and h_0 is such that $0 < h_0 \leqq 1$, then the greatest lower bound of $S(h_0)$ is attained by the lower Riemann sum for the partition

$$P^*(h_0) = (0, 1 - nh_0, 1 - (n-1)h_0, \ldots, 1 - h_0, 1) ,$$

where $n = [1/h_0]$.

Using the same notation as in the first paragraph of the proof of Theorem I and making appropriate substitutions, such as "lower" for "upper," we find, again, that the contribution of

[1]) Note that C_2 may be empty.

a rectangle with a mesh from C_1 as base is consistent with the offered conclusion of the theorem.

Let us now turn to the consideration of the contribution to the lower Riemann sum of $P(h_0)$ by a rectangle erected on a mesh that belongs to C_2. Figures II and III will aid in the analysis.

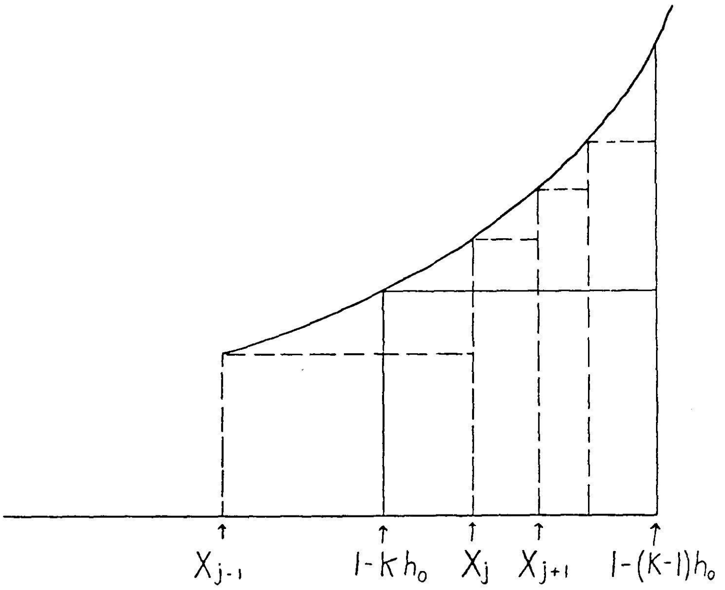

Fig. II.

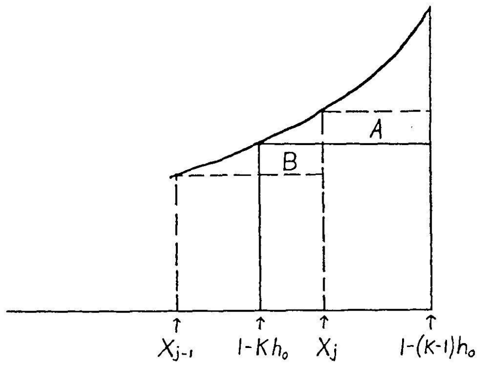

Fig. III.

Figure II indicates the typical situation at every point $1 - kh_0$ of $P^*(h_0)$ that is interior to a mesh $[x_{j-1}, x_j]$ of C_2. It is most convenient, however, to conceive of a situation such as the one indicated in Figure III in which the points of $P(h_0)$, lying between x_j and $1 - (k - 1) h_0$, are ignored [1]. This step can only cause a decrease in the lower Riemann sum for $P(h_0)$ and we will prove that even this reduced sum is greater than, or equal to, the lower Riemann sum for $P^*(h_0)$.

It will be proved that the area of the rectangle denoted by B (Figure III) is less than the area of the rectangle denoted by A. Thus, the lower Riemann sum for $P(h_0)$ " gains " more (A) than it " loses " (B). Note that

$$0 < x_j - x_{j-1} \leqq h_0 \quad \text{and} \quad x_{j-1} < 1 - kh_0 < x_j .$$

The area of rectangle B is $[x_j - (1 - kh_0)]\,[(1 - kh_0)^2 - x_{j-1}^2]$ and the area of rectangle A is

[1] Note that the rectangles generated by $P(h_0)$ that lie between x_j and $1 - (k - 1) h_0$ are replaced by a rectangle over $[x_j, 1 - (k - 1) h_0]$ with height x_j^2.

$$[\{1-(k-1)h_0\}-x_j]\,[x_j^2-(1-kh_0)^2]\,.$$

Now, since $x_j > (1 - kh_0)$, it suffices to prove that

$$(1-kh_0)^2-x_{j-1}^2<[\{1-(k-1)h_0\}-x_j]\,[x_j+(1-kh_0)]\,.$$

Simple manipulation shows that this inequality is equivalent to

$$(x_j-x_{j-1})(x_j+x_{j-1})<h_0\,[x_j+(1-kh_0)]\,.$$

But this latter inequality follows from the simultaneous inequalities $0 < x_j - x_{j-1} \leqq h_0$ and $x_{j-1} < (1 - kh_0)$ which are given. The fact that the area of rectangle B is less than the area of rectangle A now follows from a reversal of the previous steps. Combining our results, one notes that the lower Riemann sum for $P\,(h_0)$ is greater than, or equal to, the upper Riemann sum for $P^*\,(h_0)$, and thus, Theorem III is proved.

THEOREM IV. Same hypotheses as Theorem III. Then the greatest lower bound $\underline{S}\,(h_0)$ of $S\,(h_0)$ is given by the formula:

$$\underline{S}\,(h_0)=h_0\,n-h_0^2\,n\,(n+1)+\frac{h_0^3\,n}{6}\,[(n+1)(2n+1)]\,.$$

THEOREM V. $\lim\limits_{h_0\to 0}\bar{S}\,(h_0)=\lim\limits_{h_0\to 0}\underline{S}\,(h_0)=1/3=\int\limits_0^1 x^2\,dx\,.$

Now, one can reap the fruits of one's labor, namely, a visual interpretation for the limit process involved in the definition of $\int\limits_0^1 x^2\,dx$. First, let h_0 vary, and consider the graph (Figure IV) of the functions $\bar{S}\,(h)$ and $\underline{S}\,(h)$. For example, to sketch $\bar{S}\,(h)$ for $1/2 < h \leqq 1$, note that $n = 1$ and that $\bar{S}\,(h) = 1 - 2h+3h^2 - h^3$. $\bar{S}\,(h)$ and $\underline{S}\,(h)$ are continuous functions each of which is composed of an infinite number of arcs of cubic functions.

Finally, in Figure V, the graph of the single valued function $\bar{S}\,(h) - \underline{S}\,(h)$ is sketched.

This function is also continuous and piecewise cubic.

Now, one observes that

$$\lim_{h\to 0}\,[\bar{S}\,(h)-\underline{S}\,(h)]=0\,,$$

and thus, the limit involved in the definition of the Riemann integral for $y = x^2$ over [0,1] can be thought of in terms of the limit of a single-valued function. All the pedagogical devices that can be used to present the idea of the limit of a single-valued function can thus be employed in the classroom development of the more sophisticated limit that occurs in the definition of the Riemann integral of $y = x^2$ over [0,1].

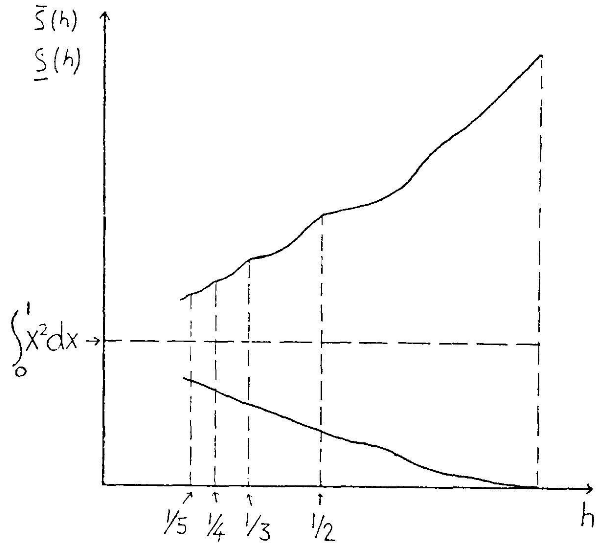

Fig. IV.

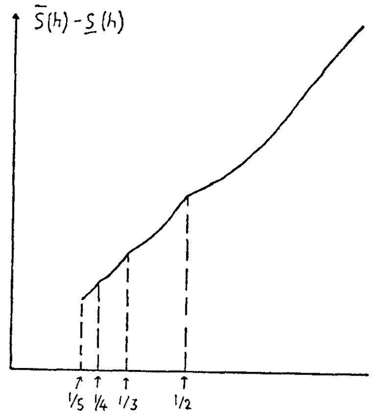

Fig. V.

Some general comments on the case of any continuous function on a closed interval seem appropriate.

(1) $\bar{S}(h_0)$ and $\underline{S}(h_0)$ always exist.

(2) $\bar{S}(h)$ is always monotone increasing (with increasing h), and $\underline{S}(h)$ is monotone decreasing (with increasing h).

(3) $S(h_0)$ attains every value c, where $\underline{S}(h_0) < c < \bar{S}(h_0)$.[1]

In conclusion, it may be noted that the graphs and analyses for the general case of a continuous function on a closed interval are similar to those for $y = x^2$ on [0,1]. In addition, the limit that occurs in the definition of the Riemann integral of such a function can also be thought of in terms of the limit of a single-valued function, namely, $\bar{S}(h) - \underline{S}(h)$.

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[1]) The question as to whether $\bar{S}(h_0)$ and $\underline{S}(h_0)$ are always attained for an arbitrary continuous function (as they were for $y = x^2$) is left open.