

5. The Theorems

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **8 (1962)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **08.08.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Let u be the center of S^{n-1} . Since f has no fixed point, it is clear that we can choose $d > 0$ so small that a closed solid n -sphere H_d^n of radius d with center at $\theta(u)$ is entirely in η^n , and H_d^n and its image $f(H_d^n)$ are contained in different half-spaces into which R^n is separated by some $(n-1)$ -plane.

Now, let S^{n-1} undergo a deformation by uniform radial shrinking toward u till it reaches a position S_2^{n-1} whose image σ_2^{n-1} under θ is contained in the interior of H_d^n . By means of θ , there results a deformation of σ^{n-1} into σ_2^{n-1} which by means of the mapping f induces a deformation, on the direction sphere, of the $(n-1)$ -cycle f^{n-1} resulting from f applied to σ^{n-1} into the $(n-1)$ -cycle f_2^{n-1} resulting from f applied to σ_2^{n-1} .

Thus the turning index of σ^{n-1} under f equals the turning index of σ_2^{n-1} under f , which by Lemma 2 equals zero. Thus Lemma 4 is proved.

5. THE THEOREMS

THEOREM 1. *Let $\eta^n \subset R^n$ be a closed n -cell and f a continuous mapping of η^n into R^n such that f maps the boundary σ^{n-1} of η^n into η^n . Then f has at least one fixed point.*

Proof. Assume no fixed points. Let, as in the case of Lemma 3, η^n and σ^{n-1} be respectively the images (under the homeomorphism θ) of the closed solid n -sphere E^n with boundary S^{n-1} , i.e., $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$.

Let u be the center of S^{n-1} . Consider the mapping f' of σ^{n-1} which maps every point $\sigma \in \sigma^{n-1}$ into the point $\theta(u)$. Since f' is the mapping which appears in the definition of the index of $\theta(u)$ relative to σ^{n-1} , we see by Lemma 3 that the turning index of σ^{n-1} under f' is non-zero.

By hypothesis, $f(\sigma) \in \eta^n$ for every $\sigma \in \sigma^{n-1}$. Hence we may deform $f(\sigma^{n-1})$ as follows. As a parameter p varies from 0 to 1, the point σ' moves in η^n along the path $\theta[\overline{\theta^{-1}f(\sigma)}, u]$ starting from σ and ending at $\theta(u)$.

For $p = 1$, the above deformation yields the mapping f' . Therefore, the $(n-1)$ -cycle resulting from f applied to σ^{n-1} is homologous on the direction sphere (as a consequence of a deformation) to the $(n-1)$ -cycle resulting from f' applied to

σ^{n-1} . Consequently, the turning index of σ^{n-1} under f equals the turning index of σ^{n-1} under f' , and hence is not zero. But this contradicts Lemma 4. Thus, Theorem 1 is true.

THEOREM 2. *Let $\eta^n \subset R^n$ be a closed n -cell with boundary σ^{n-1} and f a continuous map of η^n into R^n which leaves no point of σ^{n-1} fixed. If there exists an inner point e of η^n and an angle α with $0 \leq \alpha \leq \pi$, such that for no point $\sigma \in \sigma^{n-1}$ is α an angle from the vector $\overline{\sigma, f(\sigma)}$ to the vector $\overline{e, \sigma}$ then f leaves at least one point fixed.*

Proof. Suppose f leaves no point fixed. We shall show that under the hypotheses of Theorem 2, either

i) for no point $\sigma \in \sigma^{n-1}$ is the direction from σ to $f(\sigma)$ opposite to that from e to σ ,

or

ii) for no point $\sigma \in \sigma^{n-1}$ is the direction from σ to $f(\sigma)$ opposite to that from σ to e .

For, otherwise we would have points σ_1 and $\sigma_2 \in \sigma^{n-1}$ such that, as σ traverses a path from σ_1 to σ_2 on σ^{n-1} , the angle between $\overline{\sigma, f(\sigma)}$ and $\overline{\sigma, e}$ would change continuously from 0 to π , hence assume the value α , a contradiction.

If i) holds, we apply Lemma 1 taking the mapping g of Lemma 1 as the mapping f , and as the mapping h , we take a mapping which makes correspond to each point $\sigma \in \sigma^{n-1}$ the intersection of the half line starting at the point e and passing through the point σ , with an $(n-1)$ -sphere V^{n-1} whose center is e and which is located completely outside of σ^{n-1} . We infer by Lemma 1 that the turning indices of σ^{n-1} under f and h are equal. Since the turning index of σ^{n-1} under h clearly equals the turning index of σ^{n-1} relative to V^{n-1} , we infer from Lemma 3 that the turning index of σ^{n-1} under f is non-zero.

If ii) holds, again by Lemmas 1 and 3 the turning index of σ^{n-1} under f is non-zero. (Here, for the mapping g of Lemma 1, we again take the mapping f , and for the mapping h , we take a mapping which makes correspond to each point $\sigma \in \sigma^{n-1}$ the intersection of the half line starting at the point e and passing through the point σ , with an $(n-1)$ -sphere V^{n-1} whose center is e and which is located completely inside of σ^{n-1}).

In short, the turning index of σ^{n-1} under the assumption of the absence of fixed points is non-zero, a fact which contradicts Lemma 4. Hence f has at least one fixed point, and Theorem 2 is proved.

COROLLARY 1. *Let E^n be a closed solid n -sphere and f a continuous mapping of E^n into R^n such that f maps the boundary S^{n-1} of E^n into E^n . Then f has at least one fixed point.*

Proof. If no point of S^{n-1} is fixed, then the hypotheses of Theorem 2 are seen to be satisfied with e at the center of the sphere E^n and $\alpha = 0$.

Clearly, Corollary 1 also follows immediately from Theorem 1. Proofs of this corollary also appear in the literature ([3], page 115).

COROLLARY 2. *Let $\eta^n \subset R^n$ be a closed n -cell with boundary σ^{n-1} , and f and g two continuous maps of η^n into R^n such that for no point $\sigma \in \sigma^{n-1}$ is $f(\sigma) = g(\sigma)$. If there exists an inner point e of η^n and a constant angle β , $0 \leq \beta \leq \pi$, such that for no point $\sigma \in \sigma^{n-1}$ is β an angle between the vectors e, σ and $f(\sigma), g(\sigma)$, then there is a point $\eta_0 \in \eta^n$ such that $f(\eta_0) = g(\eta_0)$.*

Proof. Consider the map h of η^n into R^n such that for every point $\eta \in \eta^n$ the vectors $\eta, h(\eta)$ and $f(\eta), g(\eta)$ are equal. By Theorem 2, the map h has a fixed point η_0 . Consequently, $f(\eta_0) = g(\eta_0)$.

REFERENCES

1. COURANT, R. and ROBBINS, R. E., *What is Mathematics ?*, Oxford, 1941.
2. L. E. J. BROUWER, Über Abbildung von Mannigfaltigkeiten, *Math. Annalen*, Vol. 71 (1912), pp. 97-115.
3. P. S. ALEKSANDROV, *Combinatorial Topology*, Vol. 3, Graylock Press, Albany, N.Y., 1960.
4. L. S. PONTRYAGIN, *Foundations of Combinatorial Topology*, Graylock Press, Rochester, N.Y., 1952.

University of Pennsylvania

and

Queens College of the City University of New York.