5. The Theorems

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Let u be the center of S^{n-1}. Since f has no fixed point, it is clear that we can choose $d > 0$ so small that a closed solid n-sphere H_d^n of radius d with center at $\theta(u)$ is entirely in η^n, and H_d^n and its image $f(H_d^n)$ are contained in different half-spaces into which R^n is separated by some $(n-1)$-plane.

Now, let S^{n-1} undergo a deformation by uniform radial shrinking toward u till it reaches a position S_2^{n-1} whose image σ_2^{n-1} under θ is contained in the interior of H_d^n. By means of θ, there results a deformation of σ^{n-1} into σ_2^{n-1} which by means of the mapping f induces a deformation, on the direction sphere, of the $(n-1)$-cycle f^{n-1} resulting from f applied to σ^{n-1} into the $(n-1)$-cycle f_2^{n-1} resulting from f applied to σ_2^{n-1}.

Thus the turning index of σ^{n-1} under f equals the turning index of σ_2^{n-1} under f, which by Lemma 2 equals zero. Thus Lemma 4 is proved.

5. The Theorems

THEOREM 1. *Let $\eta^n \subset R^n$ be a closed n-cell and f a continuous mapping of η^n into R^n such that f maps the boundary σ^{n-1} of η^n into η^n. Then f has at least one fixed point.*

Proof. Assume no fixed points. Let, as in the case of Lemma 3, η^n and σ^{n-1} be respectively the images (under the homeomorphism θ) of the closed solid n-sphere E^n with boundary S^{n-1}, i.e., $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$.

Let u be the center of S^{n-1}. Consider the mapping f' of σ^{n-1} which maps every point $\sigma \in \sigma^{n-1}$ into the point $\theta(u)$. Since f' is the mapping which appears in the definition of the index of $\theta(u)$ relative to σ^{n-1}, we see by Lemma 3 that the turning index of σ^{n-1} under f' is non-zero.

By hypothesis, $f(\sigma) \in \eta^n$ for every $\sigma \in \sigma^{n-1}$. Hence we may deform $f(\sigma^{n-1})$ as follows. As a parameter p varies from 0 to 1, the point σ' moves in η^n along the path $\theta[\overline{\theta^{-1} f(\sigma), u}]$ starting from σ and ending at $\theta(u)$.

For $p = 1$, the above deformation yields the mapping f'. Therefore, the $(n$-$1)$-cycle resulting from f applied to σ^{n-1} is homologous on the direction sphere (as a consequence of a deformation) to the $(n-1)$-cycle resulting from f' applied to

σ^{n-1}. Consequently, the turning index of σ^{n-1} under f equals the turning index of σ^{n-1} under f', and hence is not zero. But this contradicts Lemma 4. Thus, Theorem 1 is true.

THEOREM 2. *Let $\eta^n \subset R^n$ be a closed n-cell with boundary σ^{n-1} and f a continuous map of η^n into R^n which leaves no point of σ^{n-1} fixed. If there exists an inner point e of η^n and an angle α with $0 \leq \alpha \leq \pi$, such that for no point $\sigma \varepsilon \sigma^{n-1}$ is α an angle from the vector $\overline{\sigma, f(\sigma)}$ to the vector $\overline{e, \sigma}$ then f leaves at least one point fixed.*

Proof. Suppose f leaves no point fixed. We shall show that under the hypotheses of Theorem 2, either

i) for no point $\sigma \varepsilon \sigma^{n-1}$ is the direction from σ to $f(\sigma)$ opposite to that from e to σ,

or

ii) for no point $\sigma \varepsilon \sigma^{n-1}$ is the direction from σ to $f(\sigma)$ opposite to that from σ to e.

For, otherwise we would have points σ_1 and $\sigma_2 \varepsilon \sigma^{n-1}$ such that, as σ traverses a path from σ_1 to σ_2 on σ^{n-1}, the angle between $\overline{\sigma, f(\sigma)}$ and $\overline{\sigma, e}$ would change continuously from 0 to π, hence assume the value α, a contradiction.

If i) holds, we apply Lemma 1 taking the mapping g of Lemma 1 as the mapping f, and as the mapping h, we take a mapping which makes correspond to each point $\sigma \varepsilon \sigma^{n-1}$ the intersection of the half line starting at the point e and passing through the point σ, with an $(n-1)$-sphere V^{n-1} whose center is e and which is located completely outside of σ^{n-1}. We infer by Lemma 1 that the turning indices of σ^{n-1} under f and h are equal. Since the turning index of σ^{n-1} under h clearly equals the turning index of σ^{n-1} relative to V^{n-1}, we infer from Lemma 3 that the turning index of σ^{n-1} under f is non-zero.

If ii) holds, again by Lemmas 1 and 3 the turning index of σ^{n-1} under f is non-zero. (Here, for the mapping g of Lemma 1, we again take the mapping f, and for the mapping h, we take a mapping which makes correspond to each point $\sigma \varepsilon \sigma^{n-1}$ the intersection of the half line starting at the point e and passing through the point σ, with an $(n-1)$-sphere V^{n-1} whose center is e and which is located completely inside of σ^{n-1}).

In short, the turning index of σ^{n-1} under the assumption of the absence of fixed points is non-zero, a fact which contradicts Lemma 4. Hence f has at least one fixed point, and Theorem 2 is proved.

COROLLARY 1. *Let E^n be a closed solid n-sphere and f a continuous mapping of E^n into R^n such that f maps the boundary S^{n-1} of E^n into E^n. Then f has at least one fixed point.*

Proof. If no point of S^{n-1} is fixed, then the hypotheses of Theorem 2 are seen to be satisfied with e at the center of the sphere E^n and $\alpha = 0$.

Clearly, Corollary 1 also follows immediately from Theorem 1. Proofs of this corollary also appear in the literature ([3], page 115).

COROLLARY 2. *Let $\eta^n \subset R^n$ be a closed n-cell with boundary σ^{n-1}, and f and g two continuous maps of η^n into R^n such that for no point $\sigma \varepsilon \sigma^{n-1}$ is $f(\sigma) = g(\sigma)$. If there exists an inner point e of η^n and a constant angle β, $0 \leqq \beta \leqq \pi$, such that for no point $\sigma \varepsilon \sigma^{n-1}$ is β an angle between the vectors $\overline{e, \sigma}$ and $\overline{f(\sigma), g(\sigma)}$, then there is a point $\eta_0 \varepsilon \eta^n$ such that $f(\eta_0) = g(\eta_0)$.*

Proof. Consider the map h of η^n into R^n such that for every point $\eta \varepsilon \eta^n$ the vectors $\overline{\eta, h(\eta)}$ and $\overline{f(\eta), g(\eta)}$ are equal. By Theorem 2, the map h has a fixed point η_0. Consequently, $f(\eta_0) = g(\eta_0)$.

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