**Zeitschrift:** L'Enseignement Mathématique

**Band:** 8 (1962)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

**Kapitel:** 7. ANOTHER DETERMINATION OF COEFFICIENTS.

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**DOI:** https://doi.org/10.5169/seals-37963

with each $\sigma_{j,\nu}^{(0)}(z)$ analytic, and $\sigma_{j,r}^{(0)}(z, \lambda)$ bounded. We shall show that the elements $\alpha_{j,\nu}^{(0)}(z)$ in (6. 7) may be so specified as to yield

$$\sigma_{j,\nu}^{(0)}(z) \equiv \begin{cases} 1 \text{ when } (j, \nu) = (1, 0), \quad \nu = 0, 1, 2, \ldots, (r-1) \\ 0 \text{ when } (j, \nu) \neq (1, 0). \end{cases} \tag{6, 14}$$

The effect of this will be to give the formula (6. 11) the form

$$m^*(\eta_i) = \lambda^q \left\{ v_i(z, \lambda) + \frac{1}{\lambda^r} \sum_{j=1}^{p} \lambda^{1-j} \sigma_{j,r}^{(0)}(z, \lambda) D^{j-1} v_i \right\}. \tag{6. 15}$$

## 7. Another determination of coefficients.

The dependence of the functions (6. 12) upon the unspecified ones $\alpha_{j,\nu}^{(0)}(z)$ of (6. 7) is advantageously set forth in terms of vector-matrix notation. To this end, let a column vector with the components $\varphi_i$, $i = 1, 2, \ldots, p$, be denoted by $(\varphi)$ and let the vector whose components are the terms in $1/\lambda^\nu$ of $(\varphi)$, namely with the components $\varphi_{i,\nu}$ $i = 1, 2, \ldots, p$, be denoted by $(\varphi)_\nu$. Also let $H$ designate the square matrix

$$H = \begin{bmatrix} \lambda^{-1}D & 0 & 0 & - & - & -\bar{\beta}_p \\ 1 & \lambda^{-1}D & 0 & - & - & -\bar{\beta}_{p-1} \\ 0 & 1 & \lambda^{-1}D & - & - & -\bar{\beta}_{p-2} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & - & -\bar{\beta}_1 + \lambda^{-1}D \end{bmatrix} \tag{7. 1}$$

the elements of which are in part functions of $z$ and $\lambda$, and in part the indicated differential operator. Again let $H_\nu$ designate the matrix that is obtainable from (7. 1) by replacing its elements by their terms in $1/\lambda^\nu$. The relations (6. 10) are then seen at once to take the form

$$(\alpha^{(k)}) = H(\alpha^{(k-1)}).$$

With iteration defined in the manner

$$H^{[k]}(\varphi) = H(H^{[k-1]}(\varphi)), \; H^{[1]} = H, \tag{7. 2}$$

and with $H^{[0]}$ signifying the unit matrix, it is then easily seen that

$$(\alpha^{(k)}) = H^{[k]}(\alpha^{(0)}). \tag{7.3}$$

The relation (6. 12) may thus be written in the form

$$(\sigma^{(0)}) = J(\alpha^{(0)}), \tag{7.4}$$

with $J$ standing for the matrix

$$J = \sum_{k=0}^{q} \bar{\gamma}_k H^{[q-k]}. \tag{7.5}$$

The evaluations

$$(\sigma^{(0)})_\nu = \sum_{j=0}^{v} J_j (\alpha^{(0)})_{v-j},$$

$$J_j = \sum_{k=0}^{q} \sum_{i=0}^{j} \bar{\gamma}_{k,i} H^{[q-k]}_{j-i},$$

evidently combine to yield the formula

$$(\sigma^{(0)})_\nu = \sum_{k=0}^{q} \sum_{j=0}^{\nu} \sum_{i=0}^{j} \bar{\gamma}_{k,i} H^{[q-k]}_{j-i} (\alpha^{(0)})_{\nu-j}. \tag{7.6}$$

In connection with this, certain observations are apropos. To begin with, the index value $j = 0$ implies $i = 0$, whereas by (6. 5) and (4. 1), $\bar{\gamma}_{k,0} = c_k(z)$. Further when $i = j$ the matrix $H^{[q-k]}_{j-i}$ reduces to precisely $K^{q-k}(z)$, with $K(z)$ as given in (3. 3). On the basis of these facts the equation (7. 6) may be arranged into the form

$$\sum_{k-0}^{q} c_k(z) K^{q-k}(z) (\alpha^{(0)})_\nu = (\sigma^{(0)})_\nu - \sum_{k=0}^{q} \sum_{j=1}^{\nu} \sum_{i=0}^{j} \bar{\gamma}_{k,i} H^{[q-k]}_{j-i} (\alpha^{(0)})_{\nu-j} \tag{7.7}$$

This is a vector equation for $(a^{(0)})_\nu$, which we shall consider for successive values of $\nu$, assuming that the values (6. 14) have been assigned.

When $\nu = 0$, the triple sum on the right of the equality in (7. 7) vanishes, and the right-hand member is, therefore, the vector $(\sigma^{(0)})_0$ whose first component is 1 and whose other components are 0. The equation is therefore a non-homogeneous one, and accordingly admits of an analytic solution for $(a^{(0)})_0$ provided the matrix multiplier of this vector on the left is non-singular. This condition is assured by the relation (3. 4).

Now we may proceed by induction. Assuming that the vectors $(a^{(0)})_j$ for $j = 1, 2, \ldots, (\nu - 1)$, have been determined and are analytic, the right-hand member of the equation (7. 7) is known. As in the case $\nu = 0$, so now, the equation is analytically solvable. The solutions for the successive values $\nu = 0, 1, 2, \ldots, (r-1)$, yield the coefficients (6. 7) for which the functions $\eta_i(z, \lambda)$, as given by the formulas (6. 8), fulfill the relations (6. 5).

## 8. On linear independence.

With the functions $a_j^{(0)}(z, \lambda)$ now at hand, we have at our disposal the $n$ known functions $y_j(z, \lambda)$, $j = 1, 2, \ldots, q$, which are the solutions of the differential equation (6. 3), and $\eta_i(z, \lambda)$, $i = 1, 2, \ldots, p$, which are given by the formulas (6. 8). We shall show that these functions are linearly independent.

Let the Wronskians of the entire set and of the respective sub-sets be denoted respectively by $W_n$, $W_q(y)$ and $W_p(\eta)$. If the usual form

$$W_n = \begin{vmatrix} y_1 & - - - & y_q & \eta_1 & - - - & \eta_p \\ Dy_1 & - - - & Dy_q & D\eta_1 & - - - & D\eta_q \\ - & - - - & - & - & - - - & - \\ - & - - - & - & - & - - - & - \\ D^{n-1}y_1 & - - - & D^{n-1}y_q & D^{n-1}\eta_1 & - - - & D^{n-1}\eta_p \end{vmatrix} \qquad (8.\ 1)$$

is modified by adding to each of the last $p$ rows suitable multiples of the preceding ones, the formula can be made to appear thus

$$= \begin{vmatrix} y_1 & - - - - & y_q & \eta_1 & - - - & \eta_p \\ Dy_1 & - - - - & Dy_q & D\eta_1 & - - - & D\eta_p \\ - & - - - - & - & - & - - - & - \\ D^{q-1}y_1 & - - - - & D^{q-1}y_q & D^{q-1}\eta_1 & - - - & D^{q-1}\eta_p \\ m^*(y_1) & - - - - & m^*(y_q) & m^*(\eta_1) & - - - & m^*(\eta_p) \\ Dm^*(y_1) & - - - - & - & - & - - - & - \\ - & - - - - & - & - & - - - & - \\ D^{p-1}m^*(y_1) & - - - & D^{p-1}m^*(y_q) & D^{p-1}m^*(\eta_1) & - - & D^{p-1}m^*(\eta_p) \end{vmatrix} \qquad (8.\ 2)$$