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Kapitel: 9. The related équation.

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9. THE RELATED EQUATION.

We are prepared now to make the construction toward which this entire discussion has been directed.

Consider the equation

$$L^*(u) = 0. \tag{9.1}$$

with

$$L^*(u) = \frac{1}{T} \begin{bmatrix} m^*(\eta_1) & - & - & - & - & m^*(\eta_p) & m^*(u) \\ Dm^*(\eta_1) & - & - & - & - & Dm^*(\eta_p) & Dm^*(u) \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ D^{p-1}m^*(\eta_1) & - & - & - & - & D^{p-1}m^*(\eta_p) & D^{p-1}m^*(u) \\ l^*(m^*(\eta_1)) & - & - & - & - & l^*(m^*(\eta_p)) & l^*(m^*(u)) \end{bmatrix}. \tag{9.2}$$

T being the determinant given in (8.4). This is clearly a differential equation of the n^{th} order in u , for which each one of the functions $y_j(z, \lambda)$ and $\eta_i(z, \lambda)$ is a solution. For if η_i is substituted for u two of the columns of the determinant (9.2) are the same, and if u is replaced y_j every element of the last column vanishes. Because the n solutions thus produced are linearly independent the solutions of the equation (9.1) are completely known.

The co-factor of the element $l^*(m(u))$ in the formula (9.2) is the determinant T . The expansion of the formula thus gives it the aspect

$$L^*(u) = l^*(m^*(u)) - \sum_{v=1}^p \frac{T_v}{T} D^{p-v} m^*(u), \tag{9.3}$$

where T_v is the determinant that is obtainable from the formula (8.4) by replacing its elements $D^{p-v} m^*(\eta_j)$ by $l^*(m^*(\eta_j))$.

From the formula (8.5) it is seen that

$$l^*(m^*(\eta_j)) = \lambda^n \sum_{v=1}^p \frac{\tau_v(z, \lambda)}{\lambda^r} \cdot \frac{D^{\mu-1} v_j}{\lambda^{\mu-1}} \tag{9.4}$$

with

$$\tau_v(z, \lambda) = \sum_{k=0}^p \bar{\beta}_k(z, \lambda) \sigma_{v,r}^{(p-k)}(z, \lambda). \tag{9.5}$$

The replacements which change T to T_v are thus seen to be ones which replace

$$\lambda^{n-v} \left\{ \delta_{p-v, j} + \frac{\sigma_{j, r}^{(p-v)}}{\lambda^r} \right\} \text{ by } \lambda^n \frac{\tau_v}{\lambda^r}.$$

It follows that

$$\frac{T_v}{T} = \lambda^v \frac{\theta_v(z, \lambda)}{\lambda^r},$$

with some function $\theta_v(z, \lambda)$ which is bounded over the z and λ domains. This gives to the relation (9.3) the form

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{v=1}^p \lambda^v \theta_v D^{p-v} m^*(u). \quad (9.7)$$

With the substitution of the expression for $D^{p-v} m^*(u)$, as it may be obtained from (4.3) by writing $\bar{\gamma}_{i-s}$ in the place of γ_{i-s} , it is found that

$$L^*(u) = l^*(m^*(u)) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \omega_j(z, \lambda) D^{n-j} u, \quad (9.8)$$

with

$$\omega_j(z, \lambda) = \sum_{v=1}^p \sum_{s=0}^p \lambda^{-s} \binom{p-v}{s} \theta_v D^s \bar{\gamma}_{\mu-v-s}.$$

A comparison of this with the earlier result (6.6) shows that

$$L^*(u) = L(u) - \frac{1}{\lambda^r} \sum_{j=1}^n \lambda^j \{ \varepsilon_j(z, \lambda) + \omega_j(z, \lambda) \} D^{n-j} u. \quad (9.9)$$

The equation (9.1), whose solutions are completely known, thus has coefficients which differ from those of the given equation (2.1) only by terms that are of at least the r^{th} degree in $1/\lambda$. It is, therefore, by definition, a related equation.

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