MODERN FUNDAMENTAL OPERATIONS IN AN EARLY ARABIC FORM: 'ANABĪ'S HEBREW COMMENTARY ON IBN LABBĀN'S KITĀB FĪ UṢŪL ḤISĀB AL-HIND

by Martin LEVEY and Marvin PETRUCK

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CONTENTS

1. QŪSHYAR IBN LABBĀN.

The apogee of Arabic science in the Orient came in the course of the tenth and eleventh centuries. This was especially true in mathematics. For example, special mention may be made of Abū al-Wafā' (940-ca. 997) who was one of the last great translators from the Greek and also was a commentator on Euclid, Diophantos, and Ptolemy [2]. There was also abū Kāmil Šuğā', the Egyptian calculator, who flourished ca. 900 and elaborated upon the algebra of al-Khwārizmī. It was abū Kāmil who influenced later European mathematics through Leonardo Fibonacci. Then there were Thābit b. Qurra (908-946), al-Isfahānī (end of tenth century), abū Ja 'far al-Khāzin (d. between 961 and 971), and others. It was a period when not only were the older works translated and improved upon from the Greek and Arabic but there were also many original works.

Abū al-Hasan Qūshyār ibn Labbān b. Bāshahrī al-Jīlī (971-1029) was one of this group [3]. He was a Persian and also wrote on astrology and astronomy. Qūshyār was variously cited as Qusyan, Qushiyad, Goshar, Lakusiar, and Gossar [4]. Other variants used for his name were Djabah [5], Halebi [6], or al-Kiya [7]. He evidently came from a place called Jili (Djilan), a village on the south of the Caspian Sea.

It has been stated that ibn Labbān was a Jew but there is no evidence for this [8]. He was the teacher of al-Nasawī (ca. 1030) who also wrote on mathematics [9]. Unfortunately, almost nothing has come down to us of ibn Labbān's biography. However, some of his mathematical works are extant as well as others in astrology and astronomy.

2. Extant works of ibn Labbān.

The following works of ibn Labbān are known [10]:

1. al-zīğ al-jami'.
2. kitāb al-mudkhal fī ṣinā 'at aḥkām al-nujūm;
3. kitāb al-asṭurlab wakaifīyat 'amalihī wa'tibārihī 'ala't-tamām wal-kamāl;
4. risalāt al-ab'ād wal-ajrām;
5. tağrīd uṣūl tarkīb al-juyūb;
6. kitāb fī uṣūl ḥisāb al-hind.

The last one is the subject of this work and will be described at length in the next section.

3. Kitāb fī uṣul ḥisāb al-hind and its Hebrew translation.

This treatise, " Book on the Foundations of Hindu Reckoning ", is extant in Arabic in only one manuscript [11]. There is also a Hebrew commentary [12], *'Iyyūn hā 'iqqārim.* The latter is treated in this study.

The Hebrew version gives not only much translation from the Arabic but also gives a very full explanation as well as a commentary upon the fundamental operations as given by ibn Labbān.

The Hebrew text is the work of Shālôm ben Joseph 'Anābî who lived in Constantinople. He completed this commentary sometime between 1450 and 1460. Other of his works extant are on the syllogism, the foundations of the Torah, and a commentary on the *Physics* of Aristotle [13].

The Arabic text is divided into two major books: the first is concerned with the fundamental operations using the decimal system while the other takes up the pure sexagesimal reckoning.

The Hebrew manuscript comprises the following twelve chapters: 1. numerals, 2. addition, 3. subtraction, 4. multiplication, 5. addition of the multiplication, 6. division, 7. remainder in division, 8. square root, 9. what comes from the root, 10. cube root, 11. what comes from the cube root, 12. checking by casting out nines. The first eight chapters are analogous to the first eight of the first book in the Arabic [14]. The twelfth Hebrew chapter is in the ninth and tenth (the last) sections of the first Arabic book. The subject of the tenth chapter of the Hebrew is found in the sixteenth section of the Arabic, book II. In most of the appropriate Hebrew chapters, an appendix discusses operations with the sexagesimal system. In this way, an attempt was made to cover the two books of ibn Labbān [15]. It is evident, therefore, that the integral sexagesimal system was in use by some people at this time.

The appreciation of the integral decimal system in the history of reckoning encountered quicker acceptance than has generally been supposed. This is seen in the reckoning treatise of al-Nasawī, the pupil of ibn Labbān. From the two extant texts of ibn Labbān's arithmetic, it is obvious that the author had written them in such an abbreviated style that it was difficult to understand when studied alone. Al-Nasawī's [16] text is essentially an elaboration of that of his teacher; it is very clear and practical and may be used without oral teaching.

4. Ibn Labbān's arithmetic in brief.

a) *Addition* (Chap. 2).

$$\text{Ex.}\quad \begin{array}{r} 5627 \\ \underline{482} \\ \hline \end{array}$$

The two amounts are written, like order to like, one above the other. Addition is begun on the left instead of on the right as in the present day method. In this example, the 4 is added to the 6 to give 10; a zero is set down and the 1 is carried to the 5. Next, the 8 is added to the 2 to give 10; a zero is set down and the 1 is carried to the row on the left. The addition is completed by adding the 7 to 2 and the 9 is set down in the proper place.

b) *Subtraction* (Chap. 3).

$$\text{Ex.}\quad \begin{array}{r} 5627 \\ -\ 482 \\ \hline \end{array}$$

First, begin on the left and subtract 4 from 6 to give 2; then subtract 8 from 2 to give 4; and 2 from 7 to give 5. Now correct for the borrowed 1 to give 1 instead of 2 in the third order from the right.

Mediation is considered as part of subtraction just as duplation is one of addition.

c) *Multiplication* (Chap. 4).

Set down the multiplicand and under it the multiplier, the first order of the lower under the last order of the upper.

$$\text{Ex.}\quad \begin{array}{r} 325 \\ \times\ 243 \\ \hline \end{array}$$

First, multiply 3 of the upper line, by 2 of the lower line, then 4, then 3 of the lower line. As each is multiplied, the new subproduct is formed. Each one is erased on the dust board as it is no longer needed and the new digit is substituted for it. The result in each case is placed on each order of the upper number in apposition to these upper orders. Or, it may be done in reverse, for example, 243 is multiplied by 3, then $3\times3 = 9$. The 9 is set in the place of the upper 3; then $3\times4 = 12$; a 2 is placed above the 4 and a 1 is carried. Then $3\times2 = 6$; a 6 plus 1 or 7 is placed above the lower 2. This gives

$$\begin{array}{l} 72925 \\ 243 \end{array}$$

Next, shift the orders left as shown below and multiply 243 by 2. First $2\times3 = 6$. This 6 is set in the place of the upper 2.

Then $2\times4 = 8$, and $8+9 = 17$. Set down the 7 in place of the 9 and carry the 1. Then $2\times2 = 4$; $4+2+1$ (carried) $= 7$

77765
243

Then move the upper line to the left one place. Then $5\times3 = 15$. Put the 5 in place of the 5 and carry 1; $5\times4 = 20$, $20+1+6 = 27$. Put 7 in place of 6 and carry the 2. Then $5\times2 = 10$; $10+2+7 = 19$.

78975
243

The answer is in the upper line.

d) *Division* (Chap. 6).

Set down the dividend and under it the divisor, the last order of the lower under the last order of the upper number.

Ex. 5627
243

First, find the first digit of the quotient; it is 2. Multiply the 243 by 2 and then subtract it from 562 to give 76. This is shown:

2
767
243

The lower number is shifted one to the right and this process is repeated to give 3. The final appearance of the problem is

23
38
243

or 23 and 38 as a remainder.

e) *Taking the square root* (Chap. 8) [17].

Ex. The root of the number 65342.

First, from the right, mark off the number by twos; 6 is left over

65342

The largest square to go into 6 is 4; its root is 2. Subtract the 4 from the 6 and set down a 2 in place of the 6. Then since the root is 2, put a 2 above the 6 and below it as follows:

2
25342
2

Double the lower 2, then move it one order to the right to give:

2
25342
 4

Then [18], obtain a number such that when it is multiplied by itself and then by 4, the product can be subtracted from 253. The number is 5. Write it as follows:

25
25342
 45

Now multiply 45 by 5 and subtract it from 253; 28 remains. It is written as follows:

25
2842
45

This process is continued [19].

f) *Taking the cube root* (Chap. 10 [29].

Ex. 2986100

Mark off this number by threes going from right to left; 2 remains on the left. The closest cube root to 2 is 1. Put a 1 over the 2 and under it twice. Subtract 1 from 2 to give:

```
1
2986100
1
1
```

Now, double the lowest 1 to give 2; multiply this 2 by the uppermost 1 (giving 2), and then adding the 1 of the third line. Then the 1 of the uppermost line is added to the 2 of the fourth; this gives:

```
1
1986100
 3
  3
```

Now, a number is desired which when multiplied by the lower 3 and then by itself and both these products are added, then added to the amount in the third line, then this sum is multiplied by the desired number, and the product can be subtracted from the part of the number (2986100) concerned. The number is 4—put the 4 after the lower 3 to get 34; then multipy by 4 to get 136; then add the 3 to get 436; multiply the whole thing (the 436) by 4 to get 1744. Subtract this from 2986

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 1 4
242100
436
 34
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here, $436 = 3a^2+(3a+b)\,b = 3a^2+3ab+b^2$; then $34 = 3a+b$.

Now, double the 4 in the lowest line to give 8; add 8 to the 30 of the lowest line to give 38. Now multiply 38 by 4 of the uppermost line to give 152. Add it to 436 to give 588; add the upper 4 to the 38 to give 42. Move the third line one place to the right and the fourth line two places to the right to get

```
1 4
242100
 588
   42
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In this figure, $588 = 436+152 = 3a^2+3ab+b^2+(3a+2b)\,b = 3a^2+6ab+3b^2 = 3\,(a+b)^2$; $42 = 3\,(a+b)$ [21]. Now, a third number (of the cube root) is sought as was the second; this gives 4 which is placed in the first line after the 14 and also after the 42 on the lowest line. Multiply the 424 of the lowest line by 4 to give 1696 and add to the third line to give 60496; multiply by 4 and subtract it from the remainder to give 116. This is shown in the diagram:

144 [22]
116
60496
424

Now, double the 4 on the right in the lowest line to give 8 or 428. Multiply this by the 4 on the right in the upper line to give 1712. Add this to the 60496 to give 62208; add 1 to the third line. The diagram is then:

144 = cube root;
116 = the remainder or 116 parts of 62209 according to ibn Labbān.

62209 [24]
428

5. Ibn Labbān's influence.

The fundamental operations of ibn Labbān are to be found reproduced almost exactly, although in much greater detail, in al-Nasawī. In the Arabic manuscript, there is a paragraph in which al-Nasawī remarks on his debt to his great teacher. The origin of ibn Labbān's algorisms is unknown. They are not so radically different from those of Indian sources to claim them as independent inventions. The main difference between the Indian algorisms and those of ibn Labbān seems to be in the shortened process effected by the latter. For example, $\{3a^2+(3a+b)\,b\}\,b$, in the cube root process, is calculated at one time to shorten the work instead of working out $3a^2\,b$, $3ab^2$, and

b^3 in cube root. Analogously in the square root operation, $2ab$ and b^2 are not reckoned separately but as one term $(2a+b)\,b$.

There is as yet no substantial evidence that ibn Labbān influenced his immediate successor mathematicians except through his student, al-Nasawī. However, it is important that the processes he used are very close to those used today so that ibn Labbān was certainly a transmitter of ancient arithmetic as well as a probable innovator in the improvement of the methods of arithmetic calculation.

Ibn Labbān's debt to the Indians, however, is clear from an early paragraph where he states, "Here I write what is necessary to establish for the general need and Hindu arithmetic, according to astronomy and according to other disciplines in the manner in which the general public uses it, whether according to the discipline as used for whole numbers or whether according to the general public's use in making change, or whatever number it is, or the general public's use for fractions refined in studies or the counted change, and for small change until he reaches the division of the numerical remainder, and for the division of the remainder of the remainder until all that is written of our statements comprises twelve chapters."

6. 'Anābî's terminology.

The Hebrew terminology is of interest since mathematicians and translators were still having difficulty in making up new termini technici even at the late date of the fifteenth century. 'Anābî's commentary, however, tries to elaborate on the new terms brought into the discussions. *Jadr* or *jadhr* in Arabic is equated to the Hebrew *shōresh.* Nowhere, however, does the commentator or ibn Labbān indicate the true understanding of the original meaning of this term as al-Khwārizmī, for example, knew it [27]. This is shown in an elementary explanation of 'Anābî.

"When he (ibn Labbān) says *jadhr*, he refers to that multiplied number or divided number, whichever it is. The example is 5 which is a root when it is either multiplied by itself, when we say 5×5, or multiplied by something other than itself, and in the case of division." [28]

Terms of interest include *nelām*, " to record a symbol ". It is so " because in Arabic *alama* is equivalent to *rōshem*, a sign.

The Arabic *tanṣīf*, " duplation ", is called in Hebrew *dōmeh*. The fractional portion or remainder of the quotient in Judaeo-Arabic is *alqūshūr*, in Hebrew *yitrōn*. The integral part of the quotient is in Judaeo-Arabic *sīkhakh*. The square root is *jadr* in Judaeo-Arabic, *jadhr* in Arabic, and in Hebrew *shōresh*.

In determining the square and cube roots, every second or third numeral of the number is marked off. In the case of the square root, the first numeral on the right and all of its alternates are called in Hebrew *medaberet*, in Judaeo-Arabic *minṭakh*; the next one and its alternates are called *elemet* in Hebrew, in Judaeo-Arabic *asā*. *Yitrōn ha-yitrōn* is the remainder of the remainder.

7. Arabic text.

The Hebrew commentary was compared and checked with ibn Labbān's Arabic text after the former had been studied. It is planned to publish a completely collated version of these two manuscripts. The text was found in the Aya Sofya Library in Istanbul (number 4857).

NOTES AND REFERENCES

[1] M. L. is indebted to the National Science Foundation and the National Institutes of Health for research grants which aided in the preparation of this paper. He is also indebted to the American Philosophical Society for aid in investigating the Arabic MS.

[2] Aldo Mieli, " La Science Arabe " (Leiden, 1938), p. 108.

[3] J. Lelewel, " Géographie du Moyen Age " (Bruxelles, 1852-7), I, XLVIII, III;
A. Mieli, *op. cit.*, 21; P. Luckey, " Die Rechenkunst bei Gamsīd b. Mas'ūd al-Kašī " (Wiesbaden, 1951), p. 73;
H. Suter, Die math. u. astron. d. Araber in *Abh. z. Gesch. d. math. Wiss.*, 10, 83-84 (1910); Nachträge Vol. 14, 168; C. Schoy, *Isis V*, 395;
L. Ideler, " Hand. der math. und tech. Chronol. " (Berlin, 1825-6), I, p. 263; *Zeit. d. Deut. Morgen. Ges. XXIV*, 375.

[4] M. STEINSCHNEIDER, "Die Heb. Uebersetzungen d. Mittelalters" (Berlin, 1893), 352. *Cf.* also N. KRAUSE, Stambuler Handschriften islamischer Mathematiker. Quell. u. Studien zur Gesch. d. math. B3 (Berlin, 1936), pp. 472-3.

[5] Lexicon bibliographicum et encyclopaedicum a Mustapha ben Abdallah Katib Jele bi dicto et nomine Haji Khalfa celebrato compositum, ed. latine vertit et commentario indicibusque instruxit G. Flügel (Leipzig-London, 1835), V, 82, VII, 851.

[6] Haji Khalfa, *op. cit.*, V, 142.

[7] *Cf.* Steinschneider, *op. cit.*, p. 566.

[8] J. TROEPFKE, "Gesch. d. Elementar-Mathematik" (Berlin, 1930), I, p. 82; P. LUCKEY, *op. cit.*, p. 73.

[9] *Cf.* H. SUTER, *Bibliotheca Mathematica*, III/7; F. WOEPCKE, *Journal Asiatique I*, p. 492 (1863).

[10] C. BROCKELMANN, "Gesch. d. Arab. Lit." (Leiden, 1937/43), I, 222-3, suppl. I, 397-8.

[11] Aya Sofya 4857/7.

[12] Bodleian, Oppenheim 211; *Cf.* M. STEINSCHNEIDER, *Zeit. f. Math.*, XII, 33; *Zeit. Deut. Morg. Ges.*, XXIV, 332.

[13] *Cf.* M. STEINSCHNEIDER, Heb. Ueber., p. 124; J. C. WOLFIUS, Bibliotheca Hebraica (Hamburg, 1815-33); M. STEINSCHNEIDER, Hebraische Bibliographie (Berlin, 1858-64), XVI, p. 103.

[14] LUCKEY, *op. cit.*, p. 75.

[15] Luckey evidently never saw the Hebrew manuscript but accepted Steinschneider's very brief and inadequate description in *Abhandl. zur Geschichte der Mathematik*, *3*, 109 (1880). It is now certain that Shalōm ben Joseph ʻAnābî knew both books of ibn Labbān contrary to Luckey's assumption.

[16] For al-Nasawī's (d. 1029/30) arithmetic, cf. F. WOEPCKE, Journal Asiatique, I, 489-500 (1863); *Vide* also Oskar SCHIRMER, "Studien zur Astronomer der Araber" (Erlangen, 1926), appendix by E. Wiedemann, pp. 46-8, 80-5; H. SUTER, *Bibliotheca Mathematica*, VII, 3rd series, pp. 113-9 (1906-7).

[17] Al-Nasawī changed the number to 57342.

[18] Ibn Labbān used an approximative method:

$$a^2 + r\,, \qquad a + \frac{r}{2a+1}\,.$$

To obtain a more exact answer zeroes were added in pairs to the original number and the answer was divided accordingly.

[19] Exactly the same method is carried out by al-Nasawī; *cf.* pp. 114-5 in H. SUTER, *Bibliotheca Mathematica*, VII, 3rd series (1906-7).

[20] *Cf.* *ibid.*, 115-7.

[21] *Cf.* H. SUTER, *op. cit.*, p. 116.

[22] $60496 = 3\,(a+b)^2 + 3\,(a+b)\,c + c^2$; $424 = 3\,(a+b) + c$
$4 \times 60496 = 241984$; $242100 - 241984 = 116$
$241984 = 3\,(a+b)^2\,c + 3\,(a+b)\,c^2 + c^3$.

[23] The remainder should be 116 parts of (62209+428) or

$$\frac{r}{3\,(a+b+c)^2+3\,(a+b+c)+1}.$$

This error is also in al-Nasawī. Essentially, ibn Labbān's approximation is

$$\sqrt[3]{a^3+r} \sim a+\frac{r}{3a^2+3a+1}.$$

[24] This is called the middle line in the text; al-Nasawī had the same nomenclature. *Cf.* Suter, *op. cit.*, p. 117.

[25] *Cf.* B. Datta and A. N. Singh, Hist. of Hindu Math. (Lahore, 1935); A. N. Singh, *Archeion*, XVIII, pp. 43-62 (1936).

[26] Suter, *op. cit.*, p. 118.

[27] S. Gandz, " The Origin of the Term Root ", *American Mathematical Monthly 33*, 161-5 (1926); *35*, 67-75 (1928).

[28] Introduction of MS.

Martin Levey,
Yale University
New Haven 11, Connecticut

Marvin Petruck
Dropsie College.