

# 3. The implicit function theorem.

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Condition (2.4) is satisfied if (2.3) holds in the sphere

$$S: \| u - u_0 \| \leq (1-l)^{-1} \| Vu_0 - u_0 \|. \quad (2.5)$$

Moreover,  $u$  is the limit of the sequence  $\{ u_n \}$  where

$$u_{n+1} = Vu_n, \quad n = 0, 1, 2, \dots,$$

and there results the estimate

$$\| u - u_{n+1} \| \leq l(1-l)^{-1} \| u_{n+1} - u_n \| \leq l^{n+1}(1-l)^{-1} \| u_1 - u_0 \|. \quad (2.6)$$

### 3. THE IMPLICIT FUNCTION THEOREM.

THEOREM 3.1. Let  $T^*$  be an operator with domain  $D \subset B_1$  and range in  $B_2$ , let  $S^* = S(u^*, r^*) \subset D$  and

$$T^* u^* = \theta. \quad (3.1)$$

We assume furthermore that there exists a linear operator  $K$  on  $S^*$  into  $B_2$  with the following properties:

- $\alpha)$   $K$  has a bounded inverse,  $K^{-1}$ , defined on  $B_2$  and
- $\beta)$  There exists a constant  $m < \| K^{-1} \|^{-1}$  such that

$$\| T^* v - T^* u - K(v - u) \| \leq m \| v - u \| \quad \text{for } u, v \in S^*. \quad (3.2)$$

Then there exists an  $\Omega = (u^*, r, a, b)$ -neighborhood of  $T^*$ , such that for all  $T \in \Omega$  the equation

$$Tu = \theta, \quad (3.1a)$$

has a unique solution  $u = u(T)$  in  $S(u^*, r)$ . This solution is continuous in  $T$  at  $T = T^*$  in the sense

$$\| u(T) - u^* \| \rightarrow 0 \quad \text{as} \quad \| Tu^* \| \rightarrow 0. \quad (3.3)$$

In this theorem the operators  $T$  and  $K$  need not be continuous.

*Proof.* Let  $T$  lie in a  $(u^*, r, a, b)$ -neighborhood of  $T^*$  with  $r \leq r^*$ . Then by (3.2), with  $\Delta T = T - T^*$ , we have

$$\begin{aligned} & \|Tv - Tu - K(v - u)\| \leq \|\Delta T v - \Delta T u\| \\ & + \|T^*v - T^*u - K(v - u)\| \leq (b + m) \cdot \|v - u\| \end{aligned} \quad (3.4)$$

for  $u, v \in S(u^*, r) \subset S^*$ ,

and the equation

$$u = Vu \equiv K^{-1}(K - T)u, \quad u \in S(u^*, r) = S, \quad (3.5)$$

is equivalent to (3.1 a),  $u \in S$ .

For every  $b \geq 0$  with  $l = (b + m) \|K^{-1}\| < 1$ , (3.4) yields

$$\begin{aligned} \|Vu - Vv\| &= \|K^{-1}[K(u - v) - Tu + Tv]\| \leq l \|u - v\|, \\ l &< 1, \quad \text{for } u, v \in S(u^*, r). \end{aligned}$$

If

$$\|Vu^* - u^*\| = \|K^{-1}Tu^*\| < (1 - l)r, \quad (3.6)$$

then the assumptions of the contraction mapping theorem [Section 2 f] are satisfied. Thus, under these conditions, there exists a unique solution  $u = u(T)$  in  $S$  satisfying the condition

$$\|u - u^*\| \leq (1 - l)^{-1} \|K^{-1}Tu^*\| \leq (1 - l)^{-1} \|K^{-1}\| \cdot \|Tu^*\|. \quad (3.7)$$

This implies the continuity (3.3).

The inequality (3.6) is satisfied if  $T \in \Omega$  with

$$a = [\|K^{-1}\|^{-1} - (b + m)]r.$$

This completes the proof.

This proof also gives quantitative conditions for  $r, a, b$  which are sufficient for the existence of a unique and continuous solution  $u$  of (3.1 a) in  $S(u^*, r)$ .

*Supplement.* The assertion of Theorem 3.1 is true for each  $\Omega$ -neighborhood of  $T^*$  with  $0 < r \leq r^*$  and  $a, b$  satisfying

$$a = [\|K^{-1}\|^{-1} - (b + m)]r > 0. \quad (3.8)$$

Then, for the solution  $u = u(T)$  in  $S$ , the estimate (3.7) holds.

A unique solution of (3.4 a) in  $S(u^*, r)$  also exists for such  $r$  and  $b$  if (3.6) holds, but in (3.8) the sign “ $>$ ” cannot be replaced by “ $\geq$ ”, nor can the constant  $a$  in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator  $T$  which is linear in  $S(u^*, r)$ .

#### 4. INVERSE FUNCTION THEOREMS.

Under the conditions of the implicit function Theorem 3.1, the operator  $T$  has a local inverse defined in a neighborhood of a point  $\omega_0$  for which

$$Tu = w. \tag{4.1}$$

has a solution  $u_0$ . This inverse has its range in a neighborhood of  $u_0$ . For the proof set  $T^*u = Tu - \omega_0$  in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution  $u$  of equation (4.1) for all  $\omega$  in  $B_2$  even if  $T$  is defined on the whole Banach space  $B_1$  and the conditions are satisfied at each point  $u$  of  $B_1$ .<sup>1)</sup>

However, this actually is not necessary for the existence of at least one solution  $u$  of (4.1) for all  $\omega \in B_2$  as is indicated by the following theorem.

**THEOREM 4.1.** Let the operator  $T$ , mapping a non-empty domain  $D \subset B_1$  into  $B_2$ , satisfy the following conditions:

For each  $u \in D$  there exist a sphere  $S(u, r) \subset D$ , a linear operator  $K$ , and a constant  $m$  such that the following conditions hold:

- $\alpha)$   $K$  has a bounded inverse  $K^{-1}$  on  $TS(u, r)$
- $\beta)$   $\|T\varphi - T\tilde{\varphi} - K(\varphi - \tilde{\varphi})\| \leq m \|\varphi - \tilde{\varphi}\|$  for  $\varphi, \tilde{\varphi} \in S(u, r)$
- $\gamma)$   $(\|K^{-1}\|^{-1} - m)r \geq c > 0$  where the constant  $c$  is independent of  $u \in D$ .

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1) Example:  $Tu \equiv \arctan u = w$ , with  $B_1 = B_2 = \{\text{real numbers}\}$ , is not solvable for all  $w \in B_2$ , although the conditions of Theorem 3.1 are satisfied at each point  $(u, w = \arctan u)$  for  $T^*u = Tu - w$ .