11. COMPLETELY CONTINUOUS OPERATORS, GLOBAL EXISTENCE THEOREMS USING THE SCHAUDER FIXED POINT THEOREM.

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Proof of Theorem 10.3. α. Let $u_0 \in B_1$ and $Tu_0 = (I - V)\, u_0 = w_0$. Then from Theorem 3.1 with $K = I - L$ it follows that an open neighborhood of w_0, $\| w - w_0 \| < a$, exists such that (10.3) is solvable for these w.

β. Let $\tilde{w}$ be an arbitrary point of B_1 and let u_0, w_0 be as above. Then the set Λ of all λ, for which

$$Tu = \lambda \tilde{w} + (1 - \lambda)\, w_0\,, \quad 0 \leqq \lambda \leqq 1\,,$$

is solvable, is non-void and open with respect to [0, 1] according to α.

λ. We show that Λ is also closed. Let $\lambda_n \in \Lambda$, $n = 1, 2, \ldots$, be a sequence which converges to λ^*. According to condition c the solutions u_n of $u = Vu + w_n$, $w_n = \lambda_n \tilde{w} + (1 - \lambda_n)\, w_0$, are bounded. Because of the complete continuity of V there exists a subsequence u_{n_i} such that Vu_{n_i} converges to some element s of the Banach space B_1.

Let $w^* = \lambda^* \tilde{w} + (1 - \lambda^*)\, w_0$. Then the sequence u_{n_i} converges to $u^* = s + w^*$ in norm. The element u^* is a solution of the equation $u = Vu + w^*$ since

$$\| u_{n_i} - Vu_{n_i} - w_{n_i} \| = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots,$$

and because of the continuity of the norm. Hence $\lambda^* \in \Lambda$ and, therefore, $\Lambda = [0, 1]$.

11. Completely continuous operators, global existence theorems using the Schauder fixed point theorem.

The previous theorems, even the global ones, are derived, roughly speaking, by applying neighborhood theorems and exhausting a domain on the boundary of which the assumptions fail to hold. Here the question suggets itself whether or not corresponding conditions in a shell near the boundary suffice for existence. This indeed is possible for equations

$$u = Vu\,, \tag{11.1}$$

with a completely continuous operator V. The proof of this statement uses Schauder's fixed point theorem.

THEOREM 11.1. Let V be a completely continuous operator mapping a domain $D \subset B_1$ into B_1 and having the following property.

There exist a point $u_0 \in D$ and non-negative numbers R and C such that

$$\| Vu - u_0 \| \leqq C \quad \text{for} \quad u \in \bar{S}(u_0, R) \subset D. \tag{11.2}$$

If $R < C$ let the additional condition be satisfied:

There is a number $l < 1$ such that

$$\| Vu - Vv \| \leqq l \| u - v \|, \tag{11.3}$$

holds for all u, v in the shell

$$R \leqq \| u - u_0 \| \leqq \frac{C - lR}{1 - l} = R_1 \quad \text{and} \quad \bar{S}_1 = \bar{S}(u_0, R_1) \subset D. \tag{11.4}$$

Then the equation (11.1) has at least one solution in $\| u - u_0 \| \leqq R^*$ where $R^* = R$ in the case $C \leqq R$ and $R^* = R_1$ for $C > R$.

Proof. α) If $C \leqq R$ then $V\bar{S} \subset \bar{S}$ and the fixed point theorem by Schauder [see 2 *d*] yields the existence of at least one solution $u \in \bar{S}$.

β) Now let $C > R$. Then obviously $R < C \leqq R_1$. Hence $\bar{S} \subset \bar{S}_1$. We prove $V\bar{S}_1 \subset \bar{S}_1$: Let $u \in \bar{S}_1$; then either $u \in \bar{S}$ or $u \in \bar{S}_1 - \bar{S}$. In the first case (11.2) implies $V\bar{S} \subset \bar{S}_1$. In the second case u lies in the shell (11.4). We set

$$v = tu_0 + (1 - t) u \quad \text{with} \quad t = 1 - \frac{R}{\| u - u_0 \|}.$$

It follows that $\| v - u_0 \| = R$. Therefore, by (11.2) and (11.3) we have

$$\| Vu - u_0 \| \leqq \| Vu - Vv \| + \| Vv - u_0 \| \leqq l \| u - v \| + C. \tag{11.5}$$

Furthermore

$$\| u - v \| = \| t(u_0 - u) \| = \| u - u_0 \| - R .$$

Hence by (11.4) and (11.5)

$$\| Vu - u_0 \| \leqq l \| u - u_0 \| - lR + C \leqq \frac{C - lR}{1 - l} = R_1 ,$$

i.e., again $Vu \in \bar{S}_1$. The fixed point theorem completes the proof.

In Theorem 11.1 the estimate $\| u - u_0 \| \leqq R^*$ cannot be improved, and the number R_1 must not be replaced by a smaller one. This can be easily shown.

The application of Theorem 11.1 is easiest if the completely continuous operator V is so constituted that (11.3) with $l < 1$ holds for all u, v outside a certain sphere $S(u_0, R)$.

Theorem 11.1 can be applied to operators of the form

$$Vu = u - (I - K)^{-1}(I - W)u = (I - K)^{-1}(W - K)u .$$

First of all, equation (11.6) shows that along with W and K the operator V is also completely continuous if $(I - K)^{-1}$ exists, i.e., if K has not the eigenvalue 1. From this and Theorem 11.1 there follows easily

THEOREM 11.2. The equation

$$u = Wu ,$$

with a completely continuous operator W has at least one solution if the conditions of Theorem 11.1 for the operator V in (11.6) are satisfied with a completely continuous operator K which has not the eigenvalue 1.

For this the following conditions *a)* or *b)* are sufficient:

a) Let $\| (I - K)^{-1} \| = k$. There exist non-negative numbers c, $m < k^{-1}$, and R such that

$$\| (W - K)u - (I - K)u_0 \| \leqq c \quad \text{for} \quad \| u - u_0 \| \leqq R ,$$

and either $ck \leqq R$ or, if $ck > R$, then

$$\| (W - K)v - (W - K)u \| \leqq m \| v - u \| ,$$

for all u, v in the shell

$$R \leqq \| u - u_0 \| \leqq \frac{k(c - mR)}{1 - km}.$$

b) Let $\| (I - K)^{-1} \| = k$. There exist numbers R and $m < k^{-1}$ such that

$$\| (W - K) v - (W - K) u \| \leqq m \| u - v \| \text{ if } \| u \| > R \text{ and } \| v \| > R .$$

12. Non-linear equations containing a linear completely continuous symmetric operator.

As we have seen in some previous theorems, under certain general conditions, the existence of a solution of an approximating equation or the existence of a solution at all, can fail only if there is no approximating linear operator with bounded inverse or if there is not everywhere such an operator. In the cases when the operators considered are differentiable this means that the derived linear operator does not have a bounded inverse or the derived linear equation fails to have a unique and bounded solution.[1]) It is, therefore, important to have conditions for the existence of a bounded inverse of a corresponding linear operator.

In the case of an operator $I - A$, where A is completely continuous, this is equivalent[2]) to the fact that $u = Au$ has only the solution $u = \theta$, i.e. 1 is not an eigenvalue of A. Here we deal only with such cases and assume our non-linear equation to have the form

$$u = LVu , \tag{12.1}$$

where L is a completely continuous operator and V is an (in general non-linear) operator. This is, indeed, the most usual form of non-linear equations with a completely continuous operator.

Moreover, we now consider the equation (12.1) in a Hilbert space, that is, the operator LV has its domain and range in a

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1) This is, of course, typical for the " regular case " of non-linear equations.
2) See footnote 2 on page 47.