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ARITHMETICAL NOTES, XI. SOME DIVISOR IDENTITIES

by Eckford Cohen

1. *Introduction.* In a series of papers [1, 2, 3, 4], the author has discussed arithmetical functions related to the unitary divisors d of a positive integer r, that is, divisors d of r, such that $(d, \delta) = 1$, where δ is the complementary divisor of r. It is the purpose of this note to derive in a unified manner some of the basic identities proved in these papers.

The second section is concerned with the unitary analogue $c^*(n, r)$ of Ramanujan's trigonometric sum $c(n, r)$, § 3 with the unitary analogue of Möbius inversion, and § 4 with orthogonality properties of $c^*(n, r)$.

2. *The sum* $c^*(n, r)$. We recall that for integers n, $c(n, r)$ is defined by

$$c(n, r) = \sum_{(x, r)=1} e(nx, r), \quad e(n, r) = \exp(2\pi in/r), \tag{1}$$

where the summation is over a reduced residue system (mod r). Let us define $(n, r)_*$ to be the largest unitary divisor of r which is a factor of n. Analogous to (1) we place [1, § 2]

$$c^*(n, r) = \sum_{(x, r)_*=1} e(nx, r), \tag{2}$$

where the summation is over those integers x (mod r) such that $(x, r)_* = 1$. Such a system of numbers is said to form a semi-reduced residue system (mod r).

We first express $c^*(n, r)$ in terms of $c(n, r)$. Let $\gamma(r)$ denote the largest divisor of r with no square factors other than 1.

Identity 1 ([3, (3.1)]).

$$c^*(n, r) = \sum_{\substack{d \mid r \\ \gamma(d)=\gamma(r)}} c(n, d). \tag{3}$$

Proof. Classifying the x in (2) according to their greatest common divisor with r, one obtains

$$c^*(n, r) = \sum_{\substack{d\delta = r \\ \delta \mid \frac{r}{\gamma(r)}}} \sum_{\substack{(x, r) = \delta \\ x (\mathrm{mod}\ r) \\ (x = \delta X)}} e(nx, r) = \sum_{\substack{d\delta = r \\ \delta \mid \frac{r}{\gamma(r)}}} \sum_{(X, d) = 1} e(n\delta X, r)$$

$$= \sum_{\substack{d\delta = r \\ \delta \mid \frac{r}{\gamma(r)}}} \sum_{(X, d) = 1} e(nX, d) = \sum_{\substack{d \mid r \\ \gamma(r) \mid d}} c(n, d),$$

which is the same as (3).

Let $\phi(r)$ denote the Euler ϕ-function and $\mu(r)$ the Möbius function. It is well known that

$$c(n, r) = \sum_{d \mid (n, r)} d\mu\left(\frac{r}{d}\right). \tag{4}$$

We also note that

$$\phi(r) = c(0, r) = \sum_{d\delta = r} d\mu(\delta), \ \mu(r) = c(1, r). \tag{5}$$

Definition. Place

$$\phi^*(r) = c^*(0, r), \ \mu^*(r) = c^*(1, r), \tag{6}$$

so that $\phi^*(r)$ is the number of integers in a semi-reduced residue system (mod r).

As corollaries of (3) we obtain the following two formulas, by virtue of (5).

Identity 2 (cf. [2, Lemma 3.1, $k = 1$]).

$$\phi^*(r) = \sum_{\substack{d \mid r \\ \gamma(d) = \gamma(r)}} \phi(d). \tag{7}$$

Identity 3 ([1, (2.9)], [3, (3.5)]).

$$\mu^*(r) = \sum_{\substack{d \mid r \\ \gamma(d) = \gamma(r)}} \mu(d) = \mu(\gamma(r)). \tag{8}$$

Notation. Let $d \,\|\, r$ and $d^* \delta = r$ be used to signify that d is a unitary divisor of r.

We shall need the following relation for a proof of the unitary analogue of (4).

LEMMA 1. *Let* k *be a divisor of* r. *Then*

$$\sum_{\substack{d \mid \frac{r}{k} \\ \gamma(r)=\gamma(dk)}} \mu(d) = \begin{cases} \mu^*(r/k) & \text{if } k \parallel r, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Remark 1. We first recall that

$$\sum_{d \mid r} \mu(d) = \varepsilon(r) = \begin{cases} 1\,(r = 1) \\ 0\,(r > 1), \end{cases} \tag{10}$$

and that $\mu(r) = 0$ unless r is square-free.

Proof. Let $\sum$ denote the sum in (9).

Case 1 ($k \parallel r$). In this case, by the above remark,

$$\sum = \sum_{\substack{d \mid \frac{r}{k} \\ \gamma(r)=\gamma(k)\gamma(d)}} \mu(d) = \sum_{\substack{d \mid \frac{\gamma(r)}{\gamma(k)} \\ d\gamma(k)=\gamma(r)}} \mu(d) = \mu\left(\frac{\gamma(r)}{\gamma(k)}\right) = \mu\left(\gamma\left(\frac{r}{k}\right)\right),$$

and the first part of (9) results by (8).

Case 2 ($k \nparallel r$). In this case let r_1 be the largest common unitary divisor of k and r, $r = r_1 {}^* r_2$, $k = r_1 {}^* k_2$, so that $k_2 \mid r_2$, $r_2/k_2 > 1$, and $\gamma(r_2) = \gamma(k_2) = \gamma(r_2/k_2)$. Hence by Remark 1,

$$\sum = \sum_{\substack{d \mid \frac{r_2}{k_2} \\ \gamma(r_2)=\gamma(k_2 d)}} \mu(d) = \sum_{d \mid \frac{r_2}{k_2}} \mu(d) = 0.$$

Identity 4 ([1, (2.7)]).

$$c^*(n, r) = \sum_{d \parallel (n, r)_*} d\mu^*\left(\frac{r}{d}\right). \tag{11}$$

Proof. By Identity 1, (4), and Lemma 1, one obtains

$$c^*(n, r) = \sum_{\substack{d\delta=r \\ \gamma(d)=\gamma(r)}} \sum_{D \mid (n, d)} D\mu\left(\frac{d}{D}\right) = \sum_{D \mid (n, r)} D \sum_{\substack{DE=d \\ d\delta=r \\ \gamma(r)=\gamma(d)}} \mu(E)$$

$$= \sum_{D \mid (n, r)} D \sum_{\substack{E\delta=r/D \\ \gamma(r)=\gamma(DE)}} \mu(E) = \sum_{\substack{D \mid (n,r) \\ D \parallel r}} D\mu^*\left(\frac{r}{D}\right),$$

which is the same as (11).

In the special case $n = 0$, one gets

Identity 5 ([1, (2.8)]).

$$\varnothing^*(r) = \sum_{d \| r} d\mu^*\left(\frac{r}{d}\right). \qquad (12)$$

3. *Unitary inversion.* In this section the following analogue of (10) is basic.

Identity 6 ([1, (2.5)]).

$$\sum_{d \| r} \mu^*(d) = \epsilon(r). \qquad (13)$$

Remark 2. As d ranges over the unitary divisors of r, $\gamma(d)$ ranges over all divisors of $\gamma(r)$.

Proof. By Remark 2, (8), and (10), it follows that

$$\sum_{d \| r} \mu^*(d) = \sum_{d \| r} \mu(\gamma(d)) = \sum_{d | \gamma(r)} \mu(d) = \epsilon(\gamma(r)),$$

which proves (13).

We define the unitary product $f^* g$ of two arithmetical functions f, g, with values in the complex field, by

$$f^* g = \sum_{d * \delta = r} f(d) g(\delta). \qquad (14)$$

Lemma 2. *The set of all arithmetical functions forms a semigroup* S *relative to the unitary product. The function* ϵ *defined by* (10) *is the identity of* S.

Proof. Evidently $\epsilon^* f = f^* \epsilon = f$ for every function f. Moreover, the associative law, $f_1{}^* (f_2{}^* f_3) = (f_1{}^* f_2)^* f_3$, is easily verified for arbitrary functions, f_1, f_2, f_3.

Identity 7 [Inversion formula, [1, Theorem 2.3]). *For functions* f, g *of* S,

$$f(r) = \sum_{d \| r} g(d) \rightleftarrows g(r) = \sum_{d * \delta = r} f(d) \mu^*(\delta). \qquad (15)$$

Proof. Let I denote the function of S defined to have the value 1 for all r, $I(r) \equiv 1$. Then (13) may be written $\mu^* * I = \epsilon$. Thus by Lemma 2, μ^* is invertible in S with the (unique) inverse

$(\mu^*)^{-1} = I$. Hence, $g * I = f \rightleftarrows g = f * \mu^*$, which is merely a reformulation of (15).

We note that (11) can be rewritten as

$$c^*(n, r) = \sum_{d*\delta=r} \epsilon_d(n)\,\mu^*(\delta)\,, \quad \epsilon_r(n) = \begin{cases} 1\,(r \mid n) \\ 0\,(r \nmid n). \end{cases} \tag{16}$$

Application of the inversion formula to (16) leads to

Identity 8 ([1, (2.2)]).

$$\sum_{d \| r} c^*(n, d) = \epsilon_r(n)\,. \tag{17}$$

Noting that $\epsilon_r(1) = \epsilon(r)$, the relation (17) reduces to (13) in case $n = 1$. In the case $n = 0$, we have

Identity 9 ([1, (2.4)]).

$$\sum_{d \| r} \phi^*(d) = r\,. \tag{18}$$

The latter result can be deduced independently on applying the unitary inversion formula to (12).

4. *Orthogonality properties.* In [3, Theorem 3.2] it was proved that for unitary divisors d_1, d_2 of r,

$$\sum_{n \equiv a+b \pmod r} c^*(a, d_1)\,c^*(b, d_2) = \begin{cases} rc^*(n, e) & \text{if } e = d_1 = d_2\,, \\ 0 & \text{if } d_1 \neq d_2\,, \end{cases} \tag{19}$$

the summation on the left extending over all a, b (mod r) such that $n \equiv a + b$. This result arose from an analogous relation satisfied by $c(n, r)$ [3, (1.4)] on application of (3). We deduce now a number of consequences of (19).

Letting $n = 0$ and noting that $c^*(-n, r) = c^*(n, r)$, (19) becomes.

Identity 10. *If* $d_1 \| r$, $d_2 \| r$, *then*

$$\sum_{a \pmod r} c^*(a, d_1)\,c^*(a, d_2) = \begin{cases} r\phi^*(e) & \textit{if } e = d_1 = d_2\,, \\ 0 & \textit{if } d_1 \neq d_2\,. \end{cases} \tag{20}$$

We prove next

Identity 11. *If* $d_1 \| r$, $d_2 \| r$, *then*

$$\sum_{d*\delta=r} c^*(d, d_1)\, c^*(d, d_2)\, \phi^*(\delta) = \begin{cases} r\phi^*(e) \text{ if } e = d_1 = d_2, \\ 0 \quad \text{if } \quad d_1 \neq d_2. \end{cases} \tag{21}$$

Remark 3. If $(e, r)_* = 1$, then $c^*(ne, r) = c^*(n, r)$.

Remark 4. ([4, Remark 2.1]). *If* $d \| r$, *then any semi-reduced residue system* (mod r) *contains such a system* (mod d).

Proof. Let the left member of (20) be denoted Σ. With $(a, r)_* = d$, we may write $a = dX$, $(X, r/d)_* = 1$. By Remark 4, one may suppose X (mod r/d) chosen so that $(X, r)_* = 1$. Hence, since d_1 and d_2 are unitary divisors of r, it follows by Remark 3 that

$$\sum = \sum_{\substack{d \| r \\ (X,r)_*=1 \\ X (\mathrm{mod}\, r/d)}} c^*(dX, d_1)\, c^*(dX, d_2) = \sum_{d \| r} c^*(d, d_1)\, c^*(d, d_2) \sum_{(X, r/d)_*=1} 1,$$

which is the left of (21). The proof is complete, by Identity 10.

Remark 5 ([1, Corollary 2.2.1, also cf. Lemma 6.1]). The function $\phi^*(r)$ is multiplicative.

We require a simple formula proved in [4, (2.2)], namely,

$$\phi^*(e_1)\, c^*\left(\frac{r}{e_1}, e_2\right) = \phi^*(e_2)\, c^*\left(\frac{r}{e_2}, e_1\right), \quad e_1 \| r, \; e_2 \| r. \tag{22}$$

Applying (22) to the second factor in the sum in (21), it results that

Identity 12 ([4, Lemma 2.4]). *If* $d_1 \| r$, $d_2 \| r$, *then*

$$\sum_{\delta \| r} c^*\left(\frac{r}{\delta}, d_1\right) c^*\left(\frac{r}{d_2}, \delta\right) = \begin{cases} r \text{ if } d_1 = d_2, \\ 0 \text{ if } d_1 \neq d_2. \end{cases} \tag{23}$$

Identity 13 ([3, Theorem 3.3]). *If* m *and* n *are integers, then*

$$\sum_{\delta \| r} \frac{c^*(m, \delta)\, c^*(n, \delta)}{\phi^*(\delta)} = \begin{cases} \left(\dfrac{r}{\phi^*(r)}\right) \phi^*((n, r)_*) \text{ if } (m, r)_* = (n, r)_*, \\ 0 \qquad \text{if } (m, r)_* \neq (n, r)_*. \end{cases} \tag{24}$$

Proof. Apply (22) to the first factor in the sum in (23), with $d_1 = r/(m, r)_*$, $d_2 = r/(n, r)_*$, and use Remarks 3 and 5.

The following relation results from (24) in the case $m = n$,

$$\sum_{d \| r} \frac{(c^*(n, d))^2}{\phi^*(d)} = \left(\frac{r}{\phi^*(r)}\right) \phi^*((n, r)_*) . \tag{25}$$

Further, the Inversion Theorem and Remark 5 give

$$(c^*(n, r))^2 = \sum_{d * \delta = r} d\phi^*((n, d)_*) \phi^*(\delta) \mu^*(\delta) . \tag{26}$$

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