IV. Another method in the case of divergence

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Finally assume that there exists an $x_1 \geqq x_0$ such that we have for all x, u with $x \geqq u \geqq x_1$:

$$\frac{\Psi'(x)}{\Psi'(u)} \geqq \frac{1}{\beta} \, (x \geqq u \geqq x_1) \, . \tag{26}$$

Then the series (18) is divergent.

22. Observe that the condition (26) is certainly satisfied from a certain x_1 on, if $\Psi(x)$ has a finite limit ω,

$$\Psi'(x) \to \omega < \infty \, (x \to \infty) \, . \tag{27}$$

23. *Proof of the Theorem 6.* Since x_0 can be replaced by any greater number we can assume, without loss of generality, that $x_1 = x_0$. Then we proceed as in the proof of the Theorem 4 defining $F(x)$ by (19) and obtain, as in the section 15, using (26):

$$F\left(\Psi(x)\right) \Psi'(x) = \lim_{\kappa \to \infty} f\left(\Psi(v_\kappa)\right) \Psi'(x) \geqq \frac{1}{\beta} \overline{\lim_{\kappa \to \infty}} f\left(\Psi(v_\kappa)\right) \Psi'(v_\kappa)$$
$$\geqq \overline{\lim_{\kappa \to \infty}} f(v_\kappa) \geqq F(x) \, .$$

24. We see that $F(x)$ satisfies the conditions of the Theorem 2; therefore the integral $\int^{\infty} F(x)\, dx$ is divergent and the same holds for the series $\sum^{\infty} F(\nu)$, as $F(x)$ is monotonically decreasing. But then the series (18) is also divergent since $f(x)$ is a majorant of $F(x)$. The Theorem 6 is proved.

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25. Theorem 7. *The assertion of the Theorem 4 remains valid if the assumption that $\Psi'(x)$ is monotonically increasing is replaced by the assumption that $\Psi'(x)$ is monotonically decreasing.*

26. *Proof.* Since in any case $\Psi'(x) \geqq 0$ there exists a finite ω such that

$$\Psi'(x) \downarrow \omega \quad (x \to \infty)$$

and, as in the sec. 17, we see that this limit is $\geqq 1$.

Define the a_ν as in the sec. 18. Since we can multiply $f(x)$ by any fixed constant, we can assume that we have:

$$f(x) \geqq 1 \quad (a_0 \leqq x \leqq a_1).$$

27. Denote the inverse function of $\Psi(x)$ by $\sigma(x) \equiv \sigma_1(x)$ and its iterated $\sigma(\sigma(x))$, $\sigma(\sigma(\sigma(x)))$,... by $\sigma_2(x)$, $\sigma_3(x)$,... and define a new function $F(x)$ in such a way that we have:

$$F(x) = \frac{F(\sigma(x))}{\Psi'(\sigma(x))} \quad (x \geqq a_1). \tag{28}$$

For this purpose we put:

$$F(x) = 1 \; (a_0 \leqq x < a_1), \; F(x) = \frac{1}{\Psi'(\sigma_1(x))} (a_1 \leqq x < a_2), \ldots$$

$$F(x) = \prod_{\nu=1}^{n} \frac{1}{\Psi'(\sigma_\nu(x))} (a_n \leqq x < a_{n+1}), \tag{29}$$

and (28) follows immediately.

28. From (29) we have for $x = a_n$ and $x \uparrow a_{n+1}$:

$$F(a_n) = \prod_{\nu=0}^{n-1} \frac{1}{\Psi'(a_\nu)}, \; F(a_{n+1}-0) = \prod_{\nu=1}^{n} \frac{1}{\Psi'(a_\nu)},$$

and, putting $\dfrac{1}{\Psi'(a_0)} = \sigma_0 \leqq 1$:

$$\frac{F(a_n)}{F(a_n-0)} = \sigma_0 = \frac{1}{\Psi'(a_0)}. \tag{30}$$

Since we have $\omega \geqq 1$, $\Psi'(x) \geqq 1$, we see that $\Psi(x) - x$ is non-decreasing, and therefore, the same holds for the length of the n – th interval between the a_ν, $a_{n+1} - a_n$. The number of the a_ν lying in an interval of the length 1 in the half-line $x \geqq a_0$ has a finite upper bound which may be denoted by k.

29. From (29) it follows obviously that $F(x)$ is continuous and monotonically increasing in any half-open interval $\langle a_n, a_{n+1})$. In the points a_ν we have a discontinuity if $\sigma_0 \neq 1$. We can therefore write for any $y \geqq a_0$:

$$F(y) \geqq \sigma_0^k F(x) \quad (y-1 \leqq x \leqq y). \tag{31}$$

30. Take here as y an integer $m \geqq a_0$, multiply by dx and integrate from $m - 1$ to m; we obtain

$$F(m) \geqq \sigma_0^k \int_{m-1}^{m} F(x)\, dx$$

and therefore, denoting by $n_0 - 1$ the first integer $> a_0$:

$$\sum_{v=n_0}^{n} F(v) \geqq \sigma_0^k \int_{n_0-1}^{n} F(x)\, dx\,.$$

31. On the other hand, the relation (28) can be written as

$$F\left(\Psi(x)\right) \Psi'(x) = F(x)\,, \tag{32}$$

and it follows therefore from the Theorem 2 that $\int^{\infty} F(x)\, dx$ is divergent. We see that the series $\sum^{\infty} F(\nu)$ diverges too.

32. In order to prove our Theorem it is therefore sufficient to prove that we have

$$f(x) \geqq F(x) \quad (x \geqq a_0)\,. \tag{33}$$

But this relation is evident in the interval $\langle a_0, a_1)$. Comparing (17) and (32) this inequality follows also for the interval $\langle a_1, a_2)$ and from there on by induction for any $x \geqq a_0$. The Theorem 7 is proved.

V. New conditions for the Euler-Maclaurin Theorem

33. One of the ideas underlying the proof of the Theorem 6 was the introduction of the condition (26) which is a kind of weakened monotony condition.

We give in what follows the corresponding generalisation of the Euler-Maclaurin convergence criterion, in which we try to weaken the monotony condition even more. Combining the conditions of the Theorem 8 with the assumptions of the Theorems 1 and 2 we obtain then further criteria for the convergence and divergence of the series (18).