V. New conditions for the Euler-Maclaurin Theorem

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30. Take here as y an integer $m \geqq a_0$, multiply by dx and integrate from $m - 1$ to m; we obtain

$$F(m) \geqq \sigma_0^k \int_{m-1}^{m} F(x)\, dx$$

and therefore, denoting by $n_0 - 1$ the first integer $> a_0$:

$$\sum_{\nu=n_0}^{n} F(\nu) \geqq \sigma_0^k \int_{n_0-1}^{n} F(x)\, dx .$$

31. On the other hand, the relation (28) can be written as

$$F(\Psi(x))\, \Psi'(x) = F(x), \tag{32}$$

and it follows therefore from the Theorem 2 that $\int^{\infty} F(x)\, dx$ is divergent. We see that the series $\sum^{\infty} F(\nu)$ diverges too.

32. In order to prove our Theorem it is therefore sufficient to prove that we have

$$f(x) \geqq F(x) \quad (x \geqq a_0). \tag{33}$$

But this relation is evident in the interval $\langle a_0, a_1)$. Comparing (17) and (32) this inequality follows also for the interval $\langle a_1, a_2)$ and from there on by induction for any $x \geqq a_0$. The Theorem 7 is proved.

V. New conditions for the Euler-Maclaurin Theorem

33. One of the ideas underlying the proof of the Theorem 6 was the introduction of the condition (26) which is a kind of weakened monotony condition.

We give in what follows the corresponding generalisation of the Euler-Maclaurin convergence criterion, in which we try to weaken the monotony condition even more. Combining the conditions of the Theorem 8 with the assumptions of the Theorems 1 and 2 we obtain then further criteria for the convergence and divergence of the series (18).

34. THEOREM 8. *Let γ, ε, K be fixed positive numbers, and A a fixed real number. Assume $f(x)$ non-negative and integrable in any finite subinterval of the interval $\langle x_0, \infty)$. If for any $y \geqq \mathrm{Max}\left(x_0, \frac{x_0 - A}{\gamma}\right)$ we have*

$$f(y) \leqq Kf(x) \quad (\gamma y + A \leqq x \leqq \gamma y + A + \varepsilon), \tag{34}$$

then from the convergence of the integral (2) follows the convergence of the series (18).

If for any $x \geqq \mathrm{Max}\left(x_0, \frac{x_0 + \gamma - A}{\gamma}\right)$ we have

$$f(x) \geqq Kf(y) \quad (\gamma x + A - \gamma \leqq y \leqq \gamma x + A), \tag{35}$$

then from the divergence of the integral (2) follows the divergence of the series (18).

35. *Proof.* If (34) holds we have, taking as y an integer v and integrating with respect to x from $\gamma v + A$ to $\gamma v + A + \varepsilon$:

$$f(v) \leqq \frac{K}{\varepsilon} \int_{\gamma v + A}^{\gamma v + A + \varepsilon} f(x)\,dx\,,$$

and therefore, denoting by n_0 a convenient integer, for any $n > n_0$:

$$\frac{\varepsilon}{K} \sum_{v = n_0}^{n} f(v) \leqq \sum_{v = n_0}^{n} \int_{\gamma v + A}^{\gamma v + A + \varepsilon} f(x)\,dx. \tag{36}$$

36. The limits of the integration in the right hand integrals lie here between $\gamma n_0 + A$ and $\gamma n + A + \varepsilon$.

If an x lies in one of the integration intervals in (36) we have

$$\gamma v + A \leqq x \leqq \gamma v + A + \varepsilon, \frac{x - A}{\gamma} - \frac{\varepsilon}{\gamma} \leqq v \leqq \frac{x - A}{\gamma}$$

and we see that any such x can lie at the most in $\frac{\varepsilon}{\gamma} + 1$ such intervals. The right hand expression in (36) is therefore

$$\leqq \left(\frac{\varepsilon}{\gamma} + 1\right) \int_{\gamma n_0 + A}^{\gamma n + A + \varepsilon} f(x)\,dx$$

and our assertion corresponding to the condition (34) is proved.

37. Assuming that (35) is satisfied we take x as an integer v and obtain, integrating with respect to y from $\gamma v + A - \gamma$ to $\gamma v + A$:

$$\gamma f(x) \geqq K \int_{\gamma v + A - \gamma}^{\gamma v + A} f(y)\, dy \,,$$

and therefore, for a convenient integer n_0,

$$\frac{\gamma}{K} \sum_{v=n_0}^{n} f(v) \geqq \sum_{v=n_0}^{n} \int_{\gamma v + A - \gamma}^{\gamma v + A} f(y)\, dy = \int_{\gamma n_0 + A - \gamma}^{\gamma n + A} f(y)\, dy \,.$$

From this inequality the assertion corresponding to the condition (35) follows immediately. The Theorem 8 is proved.

38. Corollary. *Assume $f(x)$ non-negative, finite and integrable in any finite subinterval of $\langle x_0, \infty)$. If there exists an integer N such that $x^N f(x)$ is from a certain x on either monotonically increasing or monotonically decreasing, the series (18) converges or diverges according as the integral (2) is convergent or divergent.*

VI. Comments on Pringsheim's discussion of the problem

39. Although Ermakof's convergence and divergence criteria and in particular Ermakof's second proof, using Abel's functional equation, are extremely interesting, they remained very little known and it appears that the author's paper [5] was the first in which the problem was taken up in a modern way. The reason for this may lie partly in the very negligent way in which Ermakof's notes were written and partly in some erroneous and misleading statements about this problem which were formulated by Pringsheim in [6], [7] and [8]. Although the essential merit of Ermakof's second paper consists just in the fact that the function $f(x)$ need not be assumed as monotonic — it is true that Ermakof does not even mention this point in [2] — Pringsheim says in [7], pp. 308-309: -"Es ist mir neuerdings gelungen, dieselben [*that is Ermakof's criteria*] von einer ihnen (auch in der von Herrn Ermakoff gegebenen Darstellung)