Section 0. Introduction

ON $L(p, q)$ SPACES [1]

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Section 0. Introduction

$L(p, q)$ spaces are function spaces which are closely related to L^p spaces. Recall that a complex-valued function f defined on a measure space (M, m) belongs to L^p if $\|f\|_p = (\int_E |f(x)|^p \, dm(x))^{1/p} < \infty$. From the definition of the above integral we have that $\|f\|_p^p$ is the least upper bound of finite sums $\Sigma y_n^p \, m(\{x \in M : y_n \leqq |f(x)| < y_{n+1}\})$ with $0 = y_1 < y_2 < \ldots$. It follows that $\|f\|_p$ is completely determined by the distribution function of f, $\lambda_f(y) = m(\{x \in M : |f(x)| > y\})$, $y > 0$. With each function $\lambda_f(y)$ we associate the function $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leqq t\}$, $t > 0$. λ_f and f^* are non-negative and non-increasing. If $\lambda_f(y)$ is continuous and strictly decreasing f^* is the inverse function of λ_f. The most important property of f^* is that it has the same distribution function as f. It follows that

$$(\int_M |f(x)|^p \, dm(x))^{1/p} = (\int_0^\infty [f^*(t)]^p \, dt)^{1/p}.$$

Let us write this equation in a more suggestive form as

$$\|f\|_p = (\frac{p}{p} \int_0^\infty [t^{1/p} f^*(t)]^p \, dt/t)^{1/p}.$$

The Lorentz space $L(p, q)$ is the collection of all f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} (\frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q \, dt/t)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leqq \infty, \quad q = \infty. \end{cases}$$

We see that $\|f\|_p = \|f\|_{pp}^*$, so $L^p = L(p, p)$. We shall see that $\|f\|_{pq_2}^* \leqq \|f\|_{pq_1}^*$, $0 < q_1 \leqq q_2 \leqq \infty$. Hence, $L(p, q_1) \subset L(p, q_2)$ for $q_1 \leqq q_2$. In particular, $L(p, q_1) \subset L^p \subset L(p, q_2) \subset L(p, \infty)$ for $0 < q_1 \leqq p \leqq q_2 \leqq \infty$.

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In this sense the $L(p, q)$ spaces give a refinement of L^p and $L(p, \infty)$. $L(p, \infty)$ plays an important role in analysis and is sometimes called weak L^p.

The fact that $L(p, q)$ space theory provides an advantageous setting for L^p theory is best seen in results concerning the Marcinkiewicz interpolation theorem. (See [32, Vol. II, p. 112].) This theorem states:

If T belongs to a certain class (quasi-linear) of operators and $\| Tf \|^*_{q_i}$, $\leqq B_i \| f \|_{p_i}$, where $1 \leqq p_i \leqq q_i \leqq \infty, i = 0, 1$, $p_0 \neq p_1$ and $q_0 \neq q_1$, then $\| Tf \|_{q_\theta} \leqq B_\theta \| f \|_{p_\theta}$, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$, $0 < \theta < 1$.

Let us weaken the hypothesis of this theorem by requiring only that $\| Tf \|^*_{q_i \infty} \leqq B_i \| f \|_{p_i 1}$, $i = 0, 1$. We can then obtain the stronger conclusion $\| Tf \|^*_{q_\theta p_\theta} \leqq B_\theta \| f \|_{p_\theta}$ as a consequence of a well known inequality of Hardy. Hence, using elementary Lorentz space theory we weaken the hypothesis, strengthen the conclusion and shorten the proof of the L^p theorem (see [15]). Also, consideration of the Lorentz space analogue (the weak type theorem of Section 3) shows that the condition $q_\theta \geqq p_\theta$ is necessary in the L^p result (see [14]).

One of the purposes of this paper is to present, in one place, the basic properties of $L(p, q)$ spaces and some tools which are useful in their study. The behavior of operators on these spaces is also studied.

For the most part, the presentation presupposes only a knowledge of basic measure theory.

Section 1 of this paper contains a development of elementary properties and inequalities which are useful in the study of Lorentz spaces. In Section 2 we develop topological properties of the spaces. $\| \quad \|^*_{pq}$ gives a natural topology for $L(p, q)$ such that $L(p, q)$ is a topological vector space. The introduction of f^{**}, an analogue of f^*, leads to a metric on $L(p, q)$. $\left(f^{**}(t) = \sup_{m(E) \geqq t} \left(\frac{1}{m(E)} \int_E |f(x)|^r \, dm(x)\right)^{1/r}, \quad 0 < r \leqq 1.\right)$ $L(p, q)$ is seen to be a Frechet space and in some cases, a Banach space. The continuity of linear, sub-linear and quasi-linear operators is considered in terms of the above mentioned metric. Continuous linear functionals on the $L(p, q)$ spaces are discussed. Section 3 is devoted to the development of two interpolation theorems for Lorentz spaces. One of these is an analogue of the Marcinkiewicz theorem on the interpolation of operators acting on L^p spaces. The other is an analogue of the Riesz-Thorin convexity theorem. (See [32, Vol. II, p. 95].) The behavior of operators on $L(p, q)$ spaces is studied in Section 4.

This is done by considering in detail some classical L^p operators. Related references are contained in Section 5.

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Section 1. ELEMENTARY PROPERTIES AND INEQUALITIES

We consider only complex-valued, measurable functions defined on a measure space (M, m). The measure m is assumed to be non-negative and totally σ-finite. We assume the functions f are finite valued a.e. and, for some $y > 0$, $m(E_y) < \infty$, where $E_y = E_y[f] = \{x \in M : |f(x)| > y\}$. As usual, we identity functions which are equal a.e.

The *distribution function of* f is defined by $\lambda(y) = \lambda_f(y) = m(E_y)$, $y > 0$. $\lambda(y)$ is non-negative, non-increasing and continuous from the right. The *non-increasing rearrangement of* f *onto* $(0, \infty)$ is defined by $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leqq t\}$, $t > 0$. Since $\lambda_f(y) < \infty$ for some $y > 0$ and f is finite valued a.e. we have that $\lambda_f(y) \to 0$ as $y \to \infty$. It follows that $f^*(t)$ is well defined for $t > 0$. $f^*(t)$ is clearly non-negative and non-increasing on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing then $f^*(t)$ is the inverse function of $\lambda_f(y)$.

It follows immediately from the definition of $f^*(t)$ that

$$f^*(\lambda_f(y)) \leqq y \,. \tag{1.1}$$

Since $\lambda_f(y)$ is continuous from the right we have

$$\lambda_f(f^*(t)) \leqq t \,. \tag{1.2}$$

Inequalities (1.1) and (1.2) can be used to prove two elementary properties of f^*.

(1.3) *$f^*(t)$ is continuous from the right.*

Proof. We have $f^*(t) \geqq f^*(t+h)$ for all $h > 0$. If there exists y such that $f^*(t) > y > f^*(t+h)$ for all $h > 0$, then, using (1.2), we have $\lambda_f(y) \leqq \lambda_f(f^*(t+h)) \leqq t + h$ for all $h > 0$. That is, $\lambda_f(y) \leqq t$. It follows that $f^*(t) \leqq y$, which is a contradiction.