Section 3. Interpolation theorems

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such that $l(f) = \int_M f(x)\, g(x)\, dm(x)$ for all $f \in L(1,1)$ and $| \int_M f(x)\, dm(x) | \leqq B \| f \|_{1q}^{*}$. If (M, m) is non-atomic we can use this to show that $g = 0$ a.e. and, hence *the trivial functional* $l \equiv 0$ *is the only continuous linear functional on* L (1, q), $1 < q < \infty$.

Section 3. Interpolation theorems

Suppose T is an operator which maps $L(p_i, q_i)$ boundedly into $L(p'_i, q'_i)$, $i = 0, 1$. An interpolation theorem for $L(p, q)$ spaces can then be described as a method which leads to inequalities of the form $\| Tf \|_{p'q'}^{*} \leqq B \| f \|_{pq}^{*}$, B independent of $f \in L(p, q)$. The intermediate spaces $L(p, q)$ and $L(p', q')$ and the corresponding constant B are determined by the method of interpolation.

Interpolation theorems can generally be classified as either weak type or strong type. The two types of theorems are easily characterized. The weak type theorems are proved by real variable methods which utilize only minimal hypotheses. Since the weak hypotheses are characteristic of the real method of proof, the conclusions are limited. In the case of Lorentz spaces the essential part of the weak type hypothesis is that the range spaces of the given end point conditions are weak L^p spaces. We can then conclude only that an intermediate space $L(p, q)$ is mapped boundedly into an appropriate space $L(p', q')$, where $q' \geqq q$. In order to utilize a stronger hypothesis to arrive at a stronger conclusion, we must go to the complex methods of proof which are characteristic of the strong type theorems. The two methods also differ in the intermediate spaces obtained and in the behavior of the corresponding constants B. In general, we obtain more intermediate spaces by the weak type methods. However, the constants corresponding to the weak type methods are, in some sense, not as satisfactory. This is seen in the prototypes of the weak and strong type theorems, the interpolation theorem of Marcinkiewicz and the Riesz-Thorin convexity theorem.

An operator T mapping functions on a measure space into functions on another measure space is called *quasi-linear* if $T(f+g)$ is defined whenever Tf and Tg are defined and if $| T(f+g) | \leqq K(| Tf | + | Tg |)$ a.e., where K is independent of f and g. An argument similar to that which led to (1.6) gives

$$(T(f+g))^{*}(t) \leqq K((Tf)^{*}(t/2) + (Tg)^{*}(t/2)). \tag{3.1}$$

Our weak type theorem is a consequence of Hardy's inequality.

WEAK TYPE THEOREM: *If* T *is quasi-linear and*

H) $\| Tf \|^*_{p'_i q'_i} \leqq B_i \| f \|^*_{p_i q_i}, \quad i = 0, 1, \quad p_o < p_1, \quad p'_o \neq p'_1,$

then

C) $$\| Tf \|^*_{p'_\theta s} \leqq B_\theta \| f \|^*_{p_\theta q},$$

where $q \leqq s$ *and, for* $0 < \theta < 1$, $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/p'_\theta = (1-\theta)/p'_0 + \theta/p'_1$. *If* $r = \min(q, q_0, q_1)$, *then* $B_\theta = O([\theta(1-\theta]^{-1/r})$.

Proof. Let $p = p_\theta$ and $p' = p'_\theta$. Since $q \leqq s$ implies $\| Tf \|^*_{p's} \leqq \| Tf \|^*_{p'q'}$, it is sufficient to prove C) with $s = q$. Similarly, we assume that $q'_0 = q'_1 = \infty$ and that $q_o, q_1 \leqq q$, except when $p_1 = q_1 = \infty$. Put

$$f^t(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| > f^*(t^\gamma) \\ 0 & \text{otherwise} \end{cases}$$

and $f_t(x) = f(x) - f(x)$, where $\gamma = \dfrac{1/p'_0 - 1/p'}{1/p_0 - 1/p} = \dfrac{1/p' - 1/p'_1}{1/p - 1/p_1}$.

It follows from the definitions that

$$(3.2) \qquad \begin{cases} f^{t*}(y) \leqq \begin{cases} f^*(y) & 0 < y < t^\gamma \\ 0 & y \geqq t^\gamma \end{cases} \\ \\ f^*_t(y) \leqq \begin{cases} f^*(t) & 0 < y < t^\gamma \\ f^*(y) & y \geqq t^\gamma . \end{cases} \end{cases} \quad and$$

Case 1: $p_1 < \infty$, $q < \infty$.

We use (3.1), a change of variables and Minkowski's inequality (or, if $q < 1$, an obvious substitute which introduces an additional factor of $2^{1/q}$) to obtain

$$\| Tf \|^*_{p'q} \leqq K 2^{1/p'} (q/p')^{1/q} \Big\{ \big(\int_0^\infty [t^{1/p'} (Tf^t)^*(t)]^q \frac{dt}{t} \big)^{1/q}$$

$$+ \big(\int_0^\infty [t^{1/p'} (Tf_t)^*(t)]^q \frac{dt}{t} \big)^{1/q} \Big\} .$$

By H), this sum is majorized by

$$K2^{1/p'}(q/p')^{1/q}\left\{\left(\int_0^\infty [B_0\, t^{1/p'-1/p_0'}\, \| f^t \|^*_{p_0q_0}]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^\infty [B_1\, t^{1/p'-1/p_1'}\, \| f_t \|^*_{p_1q_1}]^q \frac{dt}{t}\right)^{1/q}\right\}.$$

By using (3.2) and Minkowski's inequality again, we dominate this by

$$\{ K2^{1/p'}(q/p')^{1/q} \}\cdot$$

$$\left\{ B_0 \left(\int_0^\infty t^{-q(1/p_0'-1/p')}\left[\frac{q_0}{p_0}\int_0^{t^\gamma} [f^*(y)]^{q_0}\, y^{(q_0/p_0)-1}\, dy\right]^{q/q_0} \frac{dt}{t}\right)^{1/q}\right.$$

$$+ B_1 \left(\int_0^\infty t^{q(1/p'-1/p_1')}\left[\frac{q_1}{p_1}\int_{t^\gamma}^\infty [f^*(y)]^{q_1} y^{(q_1/p_1)-1}\, dy\right]^{q/q_1} \frac{dt}{t}\right)^{1/q}$$

$$\left. + B_1 \left(\int_0^\infty t^{q(1/p'-1/p_1)}\left[\frac{q_1}{p_1}\int_0^{t^\gamma} [f^*(t^\gamma)]^{q_1}\, y^{(q_1/p_1)-1}\, dy\right]^{q/q_1} \frac{dt}{t}\right)^{1/q}\right\}.$$

Again changing variables and then using Hardy's inequality, we majorize the last sum by

$$K2^{1/p'}\,|\,\gamma\,|^{-1/q}(p/p')^{1/q}\left\{\frac{B_0}{(1-(p_0/p))^{1/q_1}} + \frac{B_1}{((p_1/p)-1)^{1/q_1}} + B_1\right\}\|f\|^*_{pq}\,.$$

(Note that in order to apply Hardy's inequality it was necessary to weaken the hypothesis so that $q/q_i \geqq 1, i = 0, 1$.)

Case 2: $p_1 < \infty$, $q = \infty$.

Following the proof of case 1, we obtain

$$t^{1/p'}(Tf)^*(t) \leqq K\cdot 2^{1/p'}\left\{ B_0\, t^{1/p'-1/p_0'}\left(\frac{q_0}{p_0}\int_0^{t^\gamma} [f^*(y)]^{q_0}\, y^{(q_0/p_0)-1}\, dy\right)^{1/q_0}\right.$$

$$+ B_1\, t^{1/p'-1/p'_1}\left(\frac{q_1}{p_1}\int_{t^\gamma}^\infty [f^*(y)]^{q_1}\, y^{(q_1/p_1)-1}\, dy\right)^{1/q_1}$$

$$\left. + B_1\, t^{1/p'-1/p_1'}\left(\frac{q_1}{p_1}\int_0^{t^\gamma} [f^*(t)]^{q_1}\, y^{(q_1/p_1)-1}\, dy\right)^{1/q_1}\right\}.$$

Then, after use of the estimate $y^{1/p} f^*(y) \leqq \|f\|^*_{p\infty}$, the proof of case 2 is clear.

The remaining cases are

Case 3: $p_1 = q_1 = \infty,\ q < \infty,$
and
Case 4: $p_1 = q_1 = q = \infty.$

The proofs of these cases follows the proofs of cases 1 and 2, except we now use the estimate $\|f_t\|^*_{\infty\infty} \leqq f^*(t^\gamma)$.

An operator T which maps functions on a measure space into functions on another measure space is called *sublinear* if whenever Tf and Tg are defined and c is a constant, then $T(f+g)$ and $T(cf)$ are defined with

$$(3.3)\qquad \begin{cases} |T(f+g)| \leqq |Tf| + |Tg| \quad \text{and} \\ |T(cf)| = |c| \cdot |Tf|. \end{cases}$$

It follows that

$$(3.4)\qquad ||Tf| - |Tg|| \leqq |T(f-g)|.$$

Our analogue of the Riesz-Thorin convexity theorem depends on harmonic majorization of subharmonic functions.

STRONG TYPE THEOREM: *Suppose* T *is a sublinear operator and*

$$\| Tf \|_{p_i' q_i'} \leqq B_i \|f\|^*_{p_i q_i}, \qquad i = 0, 1.$$

Then $\| \mathrm{Tf} \|^*_{p_\theta' q_\theta'} \leqq \mathrm{B}\mathrm{B}_0^{1-\theta}\, \mathrm{B}_1^\theta \,\| \mathrm{f} \|^*_{p_\theta q_\theta}$, *where* $1/\mathrm{p}_\theta = (1-\theta)/\mathrm{p}_0 + \theta/\mathrm{p}_1$, $1/\mathrm{p}_\theta' = (1-\theta)/\mathrm{p}_0' + \theta/\mathrm{p}_1'$, $1/\mathrm{q}_\theta = (1-\theta)/q_0 + \theta/\mathrm{q}_1$ and $1/\mathrm{q}_\theta' = (1-\theta)/\mathrm{q}_0' + \theta/\mathrm{q}_1'$, $0 < \theta < 1$.

Proof. Let $p_\theta = p$, $q_\theta = q$, $p_\theta' = p'$, and $q_\theta' = q'$.

Suppose that f is a simple function. Then f can be written in the form

$$f(x) = e^{i \arg f(x)} (G_0(x))^{1-\theta} (G_1(x))^\theta,$$

where G_i is a non-negative simple function such that

$$(3.5)\qquad \| G_i \|^*_{p_i q_i} \leqq B(\|f\|^*_{pq})^{q/q_i}, \qquad i = 0, 1.$$

To see this, consider $(f^*)^{**}$, $0 < r < \min(p_0, p_1, q_0, q_1, q_0', q_1')$. We have $(f^*)^{**}(t) = (h_0(t))^{1-\theta}(h_1(t))^\theta$, where

$$h_i(t) = [(f^*)^{**}(t)]^{q/q_i}\, t^{(q/q_i)(q/p - q_i/p_i)}, \qquad i = 0, 1.$$

If $Sh(u) = (\int_u^\infty [h(t)]^r \frac{dt}{t})^{1/r}$, it is not difficult to see that $f^*(u) \leqq S((f^*)^{**})(u)$, and hence, by Holder's inequality, that $f^*(u) \leqq (Sh_0(u))^{1-\theta} (Sh_1(u))^\theta$. G_i is obtained by choosing values smaller than Sh_i. (3.5) follows from Hardy's inequality.

Let $F(x, z) = e^{i \arg f(x)} [G_0(x)]^{1-z} [G_1(x)]^z$, z complex, $0 \leqq Re\, z \leqq 1$.

Since G_i is simple and non-negative, $i = 0, 1$, $TF(\ , z)$ is defined for z fixed. By considering first a countable dense set $\{z_k\}_{k \geqq 1}$ and then.extending by continuity to all z, we may assume that except for a set of measure zero $| TF(y, z) |$ is defined for all z and y fixed and (3.3) and (.3.4) are true pointwise in y. Fix such a point y. (3.3) and (3.4) imply that $| TF(y, z) |$ is a bounded and continuous function of z, $0 \leqq Re\, z \leqq 1$. We need that $\log | TF(y, z) |$ is subharmonic in $0 < Re\, z < 1$. This follows from the fact that $|TF(y, z)| e^{h(z)}$ is subharmonic for every harmonic function $h(z)$. That is, let $H(z)$ be analytic with real part $h(z)$. For a fixed point z let z_{km}, $k = 1, \ldots, m$, be points which are evenly distributed over the circle with radius r and center z, $m \geqq 1$. If $D(x, m, z)$ is defined by

$$e^{H(z)} F(x, z) = \frac{1}{m} \sum_{k=1}^{m} F(x, z_{km}) e^{H(zkm)} + D(x, m, z),$$

then

$$e^{h(z)} | TF(y, z) | \leqq \frac{1}{m} \sum_{k=1}^{m} e^{h(zkm)} | TF(x, z_{km}) | + | TD(y, m, z) | .$$

Since $D(x, m, z)$ is of the form $\sum_{j=1}^{N} (\varphi_j(z) - \frac{1}{n} \sum_{k=1}^{m} \varphi_j(z_{km})) \chi_{E_j}(x)$, with φ_j analytic, we may again assume that (3.3) holds pointwise in y, so $| TD(y, m, z) | \to 0$ as $m \to \infty$. Then

$$e^{h(z)} | TF(y, z) | \leqq \frac{1}{2\pi} \int_0^{2\pi} e^{h(z+re^{i\theta})} | TF(y, z+re^{i\theta}) | d\theta ,$$

so $\log | TF(v, z) |$ is subharmonic.

The preceding paragraph implies that $\log | TF(y, z) |$ is majorized in $0 < Re\, z < 1$ by the Poisson integral of its boundary values. In particular,

$\log | TF(y,\theta) | \leqq \int_{-\infty}^{\infty} P_0(\theta,t) \log | TF(y, it) | dt + \int_{-\infty}^{\infty} P_1(\theta,t) \log | TF(y,$

$1+it) \mid dt$, where $P_0(\theta, t)$ and $P_1(\theta\, t)$ and $P_1(\theta, t)$ are positive, $\int_{-\infty}^{\infty} P_0(\theta, t)\, dt = 1-\theta$ and $\int_{-\infty}^{\infty} P_1(\theta, t)\, dt = \theta$. We then obtain

$$|TF(y, \theta)|^r \leqq \left\{\exp\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t)\log|TF(y, it)|^r\, dt\right)\right\}^{1-\theta}$$

$$\cdot\left\{\exp\left(\frac{1}{\theta}\int_{-\infty}^{\infty} P_1(\theta, t)\log|TF(y, 1+it)|^r\, dt\right)\right\}^{\theta}.$$

Noting that $TF(y, \theta) = Tf(y)$, we use Jensen's inequality to obtain

$$|Tf(y)| \leqq H_0(y)^{1-\theta}\cdot H_1(y)^{\theta},$$

where

$$H_0(y) = \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t)\,|TF(y, it)|^r\, dt\right)^{1/r}$$

and

$$H_1(y) = \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_1(\theta, t)\,|TF(y, 1+it)/^r\, dt\right)^{1/r}.$$

Holder's inequality implies $(Tf)^{**}(y) \leqq \{H_0^{**}(y)\}^{1-\theta}\{H_1^{**}(y)\}^{\theta}$ and then $\|Tf\|^*_{p'q'} \leqq B\,\|H_0\|^{1-\theta}_{p_0'q_0'}\,\|H_1\|^{\theta}_{p_1'q_1'}$.

By Fubini's theorem, $H_0^{**}(y) \leqq \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P(\theta, t)\,[TF^{**}(y, it)]^r dt\right)^{1/r}$.

Hence

$$\|H_0\|_{p_0'q_0'} \leqq \left(\frac{q_0'}{p_0'}\int_0^{\infty}\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta,t)\,[TF^{**}(y,it)]^r dt\right)^{q_0/r} y^{(q_0'/p_0')-1} dy\right)^{1/q}.$$

By Jensen's inequality the right hand term is dominated by

$$\left(\frac{q_0'}{p_0'}\int_0^{\infty}\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t)\,[TF^{**}(y, it)]^{q_0'} dt\right) y^{(q_0'/p_0')-1}\, dy\right)^{1/q_0'}.$$

Thus, using Fubini's theorem, our hypothesis and (3.5), we have

$$\|H_0\|_{p_0'q_0'} \leqq \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t)\,\|TF(\cdot, it)\|^{q_0'}_{p_0'q_0'}\, dt\right)^{1/q_0'}$$

$$\leqq BB_0 \Big(\frac{1}{1-\theta} \int_{-\infty}^{\infty} P_0(\theta, t) \, \| F(\cdot, it) \|_{p_0 q_0}^{q_0'} dt \Big)^{1/q_0'}$$

$$= BB_0 \, \| G_0 \|_{p_0 q_0}^{*} \leqq BB_0 \, [\| f \|_{pq}^{*}]^{q/q_0} .$$

Similarly, $\| H_1 \|_{p_1' q_1'} \leqq BB_1 \, [\| f \|_{pq}^{*}]^{q/q_1}$.

We now have

$$\| Tf \|_{p'q'}^{*} \leqq BB_0^{1-\theta} B_1^{\theta} \, \| f \|_{pq}^{*} \tag{3.6}$$

where f is any simple function.

For any $f \in L(p, q)$ we find a sequence of simple functions f_n such that $\| f_n \|_{pq} \to \| f \|_{pq}$ and $| Tf_n | \to | Tf |$ a.e. Then, using Fatou's lemma, we have $(Tf)^{**}(t) \leqq \liminf (Tf_n)^{**}(t)$ and $\| Tf \|_{p'q'} \leqq \liminf \| TF_n \|_{p'q'}$. (3.6) then implies that $\| Tf \|_{p'q'}^{*} \leqq BB_0^{1-\theta} B_1^{\theta} \| f \|_{pq}^{*}$.

Note that in case $p_0 = q_0$, $p_1 = q_1$, $p_0' = q_0'$ and $p_1' = q_1'$ the proof is simpler and the constant B may be omitted from the conclusion so the constant $B_0^{1-\theta} B_1^{\theta}$ of the Riesz-Thorin convexity theorem is retained.

Section 4. Applications

Many classical operators are known to map L^p boundedly into $L^{p'}$, where the points $(1/p, 1/p')$ form a non-degenerate line segment and $p \leqq p'$. Operators of this type are, for example, the Fourier transform [32, Vol. I, p. 254], the Hilbert transform [23], the Hardy-Littlewood maximal function operator [32, Vol. I, p. 32], singular integral operators [4] and fractional integral operators [12] and [28]. We see from the weak type interpolation theorem that operators of this type map $L(p, q)$ boundedly into $L(p', q')$, $0 < q \leqq \infty$. Hence, we know the behavior of the operators acting on some additional spaces. If $p = p'$, this is the only extension of the L^p results. However, if $p < p'$, the L^p result is improved, since we see that L^p is mapped boundedly into $L(p', p)$, a space which is continuously contained in $L^{p'}$.

The germ of the weak type theorem can be seen in a theorem of Hardy and Littlewood on the rearrangement of Fourier coefficients. (See [32, Vol. II, p. 130].) Let us develop an $L(p, q)$ version of this result for the Fourier integral transform.