Section 4. Applications

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **12 (1966)**

Heft 4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **01.06.2024**

$$\leqq BB_0 \Big(\frac{1}{1-\theta} \int_{-\infty}^{\infty} P_0(\theta, t) \, \| F(\cdot, it) \|_{p_0 q_0}^{q_0'} \, dt \Big)^{1/q_0'}$$

$$= BB_0 \, \| G_0 \|_{p_0 q_0}^{*} \leqq BB_0 \, [\, \| f \|_{pq}^{*}]^{q/q_0} .$$

Similarly, $\| H_1 \|_{p_1' q_1'} \leqq BB_1 \, [\| f \|_{pq}^{*}]^{q/q_1}$.

We now have

$$\| Tf \|_{p'q'}^{*} \leqq BB_0^{1-\theta} B_1^{\theta} \, \| f \|_{pq}^{*} \tag{3.6}$$

where f is any simple function.

For any $f \in L(p, q)$ we find a sequence of simple functions f_n such that $\| f_n \|_{pq} \to \| f \|_{pq}$ and $| Tf_n | \to | Tf |$ a.e. Then, using Fatou's lemma, we have $(Tf)^{**}(t) \leqq \liminf (Tf_n)^{**}(t)$ and $\| Tf \|_{p'q'} \leqq \liminf \| TF_n \|_{p'q'}$. (3.6) then implies that $\| Tf \|_{p'q'}^{*} \leqq BB_0^{1-\theta} B_1^{\theta} \| f \|_{pq}^{*}$.

Note that in case $p_0 = q_0$, $p_1 = q_1$, $p_0' = q_0'$ and $p_1' = q_1'$ the proof is simpler and the constant B may be omitted from the conclusion so the constant $B_0^{1-\theta} B_1^{\theta}$ of the Riesz-Thorin convexity theorem is retained.

Section 4. Applications

Many classical operators are known to map L^p boundedly into $L^{p'}$, where the points $(1/p, 1/p')$ form a non-degenerate line segment and $p \leqq p'$. Operators of this type are, for example, the Fourier transform [32, Vol. I, p. 254], the Hilbert transform [23], the Hardy-Littlewood maximal function operator [32, Vol. I, p. 32], singular integral operators [4] and fractional integral operators [12] and [28]. We see from the weak type interpolation theorem that operators of this type map $L(p, q)$ boundedly into $L(p', q')$, $0 < q \leqq \infty$. Hence, we know the behavior of the operators acting on some additional spaces. If $p = p'$, this is the only extension of the L^p results. However, if $p < p'$, the L^p result is improved, since we see that L^p is mapped boundedly into $L(p', p)$, a space which is continuously contained in $L^{p'}$.

The germ of the weak type theorem can be seen in a theorem of Hardy and Littlewood on the rearrangement of Fourier coefficients. (See [32, Vol. II, p. 130].) Let us develop an $L(p, q)$ version of this result for the Fourier integral transform.

We write the Fourier transform of a function $f \in L^1(E_n)$ as

$$f^{\wedge}(x) = \int_{E_n} f(y)\, e^{-2\pi i x \cdot y}\, dy\,. \tag{4.1}$$

Recall that if s is a simple function then (4.1) defines $s^{\wedge}$ and we have $\| s^{\wedge} \|_2 = \| s \|_2$. The Fourier transform can then be uniquely extended such that $\| f^{\wedge} \|_2 = \| f \|_2$ for all $f \in L^2(E_n)$. Suppose $f \in L(p, q)$, $1 < p < 2$, $1 \leqq q \leqq \infty$. Then $f \in L^1 + L^2$ and, hence, $f^{\wedge}$ is defined. We have (4.1) and

$$f(x) = \int_{E_n} f^{\wedge}(y)\, e^{2\pi i x \cdot y}\, dy = (f^{\wedge})^{\vee}(x)\,, \tag{4.2}$$

in the sense that

$$\int_{|y| \leqq R} f(y)\, e^{-2\pi i x \cdot y}\, dy \to f^{\wedge}(x) \quad \text{and} \quad \int_{|x| \leqq R} f^{\wedge}(x)\, e^{2\pi i x \cdot y}\, dx \to f(y)$$

in the appropriate $L(p, q)$ norm as $R \to \infty$.

THEOREM 4.3 *Suppose* $1 \leqq q \leqq \infty$, $1 < p < 2$ *and* $1/\mathrm{p} + 1/\mathrm{p}' = 1$.

(a) $\mathrm{f} \in L(\mathrm{p}, \mathrm{q})$ *if and only if for all* F *such that* $\mathrm{F}^* = f^*$, *there exists* $\mathrm{F}^{\wedge} = g \in \mathrm{L}(\mathrm{p}', \mathrm{q})$. *Furthermore,* $\mathrm{g}^{\wedge} = \mathrm{F}$ a.e. *and* $\| \mathrm{g} \|^*_{\mathrm{p}'q} \leqq \mathrm{B} \| f \|^*_{pq}$;

(b) $\mathrm{g} \in \mathrm{L}(\mathrm{p}', \mathrm{q})$ *if and only if, for some* G *such that* $\mathrm{G}^* = \mathrm{g}^*$, *there exists* $\mathrm{G}^{\vee} = \mathrm{f} \in \mathrm{L}(\mathrm{p},\mathrm{q})$. *Furthermore,* $\mathrm{f}^{\vee} = \mathrm{G}$ q.e. *and* $\| \mathrm{g} \|^*_{p'q} \leqq \mathrm{B} \| f \|^*_{pq}$.

The proof of Theorem 4.3 depends on a result which is a slight extension of a lemma found in [32, Vol. II, p. 129]:

LEMMA 4.4. *Suppose* $\mathrm{f}(\mathrm{t})$ *is non-negative, locally integrable and an even function of* t, $-\infty < \mathrm{t} < \infty$. *Further, suppose* $\mathrm{f}(\mathrm{t})$ *is non-increasing on* $(0, \infty)$ *and* $\mathrm{f}(\mathrm{t}) \to 0$ *as* $\mathrm{t} \to \infty$. *Then* $\mathrm{g}(\mathrm{x}) = \int_0^\infty \mathrm{f}(\mathrm{t}) \cos \mathrm{xt}\, \mathrm{dt} \in \mathrm{L}(\mathrm{r}, \mathrm{q})$ *if and only if* $\mathrm{f} \in \mathrm{L}(\mathrm{r}', \mathrm{q})$, *where* $1 \leqq \mathrm{q} \leqq \infty$, $1 < \mathrm{r} < \infty$ *and* $1/\mathrm{r} + 1/\mathrm{r}' = 1$.

Proof. Suppose $g \in L(r, q)$. Let $G(x) = \int_0^x g(y)\, dy$. Then $| G(x) | \leqq | x |\, g^{**}(| x |)$. Elementary arguments show that

$$G(x) = \int_0^\infty f(t) \sin xt \frac{dt}{t}\,,$$

and then $| G(x) | \geqq B f(1/| x |)$. (See [32, Vol. II, p. 129].) It follows that

$f^*(t) \leqq B(1/t)\, g^{**}(1/t)$, $t > 0$. A change of variables and Hardy's inequality then show that $\|f\|^*_{r'q} \leqq B\,\|g\|^*_{rq}$.

Conversely, suppose $f \in L(r', q)$. We have

$$|g(x)| \leqq B \int_0^{1/|x|} f(y)\, dy\,.$$

(See [32, Vol. II, p. 129].) Hence, $|g(x)|$ is majorized by $(1/|x|) f^{**}(1/|x|)$. It follows that $g^*(t) \leqq B(1/t) f^{**}(1/t)$, $t > 0$. As above, this implies that $\|g\|^*_{rq} \leqq B\,\|f\|^*_{r'q}$.

Proof of Theorem 4.3. $F^\wedge$ and $G^\vee$ are given by (4.1) and (4.2). The inequalities are obtained from the weak type interpolation theorem and the end point results $\|f^\wedge\|_2 = \|f\|_2$ and $\|f^\wedge\|_\infty \leqq \|f\|_1$. The theorem is then clear for $n = 1$, since $2g = f^\wedge = f^\vee$ for functions of the type described in Lemma 4.4. For $n > 1$ use special functions of the form

$$f(x_1) \cdot \chi_{[0,1]}(x_2) \ldots \chi_{[0,1]}(x_n)\,,$$

where $x = (x_1, \ldots, x_n)$ and f is as in Lemma 4.4.

We prove a multiplication theorem for functions belonging to $L(p, q)$ spaces. This result is used to prove a convolution theorem for the $L(p, q)$ spaces which are Banach spaces. Note that functions which do not belong to one of the Banach spaces are not appropriate for convolution since they are not necessarily locally integrable.

THEOREM 4.5. (Multiplication theorem):

$$\|fg\|^*_{pq} \leqq B\,\|f\|_{p_0 q_0}\,\|g\|_{p_1 q_1}\,,$$

where $1/p = 1/p_0 + 1/p_1$ *and* $1/q = 1/q_0 + 1/q_1$.

Proof. Applying Holder's inequality twice, we obtain $(fg)^{**}(t, r) \leqq f^{**}(t, 2r)\, g^{**}(t, 2r)$ and then the theorem.

Suppose (G, dm) is a locally compact unimodular topological group, where dm is Haar measure on the group G. The convolution of two functions is then defined by $f * g(x) = \int f(y)\, g(xy^{-1})\, dm(y)$, provided the integral exists. We develop a convolution theorem for $L(p, q)$ spaces by interpolating certain end point results.

LEMMA 4.6. $\|f * g\|^*_{p_1 \infty} \leqq B\,\|f\|^*_{11}\,\|g\|^*_{p_1\infty}$, $1 < p_1 < \infty$.

Proof. $(f * g)^*(t) \leqq \sup\limits_{m(E) \geqq t} \dfrac{1}{m(E)} \int_E |f * g(x)|\, dm(x)\,.$

By Fubini's theorem

$$\frac{1}{m(E)}\int_E |f^*g(x)|\,dm(x) \leqq \frac{1}{m(E)}\int_E \left(\int |f(y)|\cdot|g(xy^{-1})|\,dm(y)\right)dm(x)$$

$$= \int\left(\frac{1}{m(E)}\int_E |g(xy^{-1})|\,dm(x)\right)|f(y)|\,dm(y),$$

but

$$\frac{1}{m(E)}\int_E |g(xy^{-1})|\,dm(x) \leqq g^{**}(t) \leqq Bt^{-1/p_1}\,||g||^*_{p_1\infty}.$$

It follows that $t^{1/p_1}(f^*g)^*(t) \leqq B\,||f||^*_{11}\cdot||g||^*_{p_1\infty}$.

LEMMA 4.7. $||f^*g||^*_{\infty\infty} \leqq B\,||f||^*_{p_1'1}\,||g||^*_{p_1\infty}$, *where* $1 < p_1 < \infty$ *and* $1/p_1 + 1/p_1' = 1$.

Proof. $|f^*g(x)| = |\int f(y)\,g(xy^{-1})\,dm(y)|$. By (1.9) this is majorized by $\int_0^\infty f^*(t)\,g^*(t)\,dt$, which is dominated by $||g||^*_{p_1\infty}\int_0^\infty f^*(t)^{-1/p_1}\,dt$.

By applying the weak type interpolation theorem to the end point results of Lemma 4.6 and Lemma 4.7 we obtain

LEMMA 4.8. $||f^*g||^*_{pq} \leqq B\,||f||^*_{p_0q}\,||g||^*_{p_1\infty}$, *where* $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1 \leqq q \leqq \infty$ *and* $1 < p_0, p_1 < \infty$.

Lemma 4.8 contains the fractional integration theorem of Hardy and Littlewood [12] and Stein and Weiss [20]. It is interesting to note that it is not true that

$$(*)\qquad ||f^*g||^*_{\infty\infty} \leqq B\,||f||^*_{p_0p_0}\,||g||^*_{p_1\infty},$$

B independent of f and g, $1/p_0 + 1/p_1 = 1$. (See [12].) Hence, the classical Maerinkiewicz interpolation theorem for L^p spaces does not apply directly to obtain Lemma 4.8. The Stein-Weiss extension of the Marcinkiewicz theorem does apply directly. (See [30].) Their theorem uses the end point result that (*) is true if f is restricted to the class of characteristic functions of measurable sets of finite measure. According to (2.5) this is equivalent to the end point result of Lemma 4.7.

LEMMA 4.9. $||f^*g||^*_{p1} \leqq B\,||f||^*_{p_0q_0}\,||g||^*_{p_1q_1}$, *where* $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1/q_0 + 1/q_1 = 1$ *and* $1 < p_0, p_1 < \infty$.

Proof. From (2.7), we have

$$\|f * g\|_{p1}^{*} = \sup_{h} B \left| \int f * g(x)\, h(x)\, dm(x) \right|,$$

where $h^{*}(t) \leqq t^{(1/p)-1}$. Let $I(h) = \int f * g(x)\, h(x)\, dm(x)$.

$$|I(h)| \leqq \int \left(\int |f(y)| \cdot |g(xy^{-1})|\, dm(y) \,|h(x)|\, dm(x) \right.$$
$$= \int |f(y)| \left(\int |g(xy^{-1})| \cdot |h(x)|\, dm(x) \right) dm(y).$$

Hence, $|I(h)| \leqq \|fk\|_{11}^{*}$, where $k(y) = \int |g(xy^{-1})| . |h(x)|\, dm(x)$. By the multiplication theorem it follows that $|I(h)| \leqq B \|f\|_{p_0 q_0}^{*} \|k\|_{p_0' q_1}^{*}$, where $1/p_0 + 1/p'_0 = 1$. But $k = |\bar{g}| * |h|$, where $\bar{g}(x) = g(x^{-1})$. Hence by Lemma 4.8,

$$\|k\|_{p_0' q_1}^{*} \leqq B \|\bar{g}\|_{p_1 q_1}^{*} \|h\|_{p' \infty}^{*} \leqq B \|\bar{g}\|_{p_1 q_1}^{*}.$$

Since (G, dm) is unimodular, we have $(\bar{g})^{*}(t) = g^{*}(t)$ and the lemma follows.

By applying the strong type interpolation theorem to the end point results of Lemma 4.8 and Lemma 4.9, we obtain

THEOREM 4.10. (Convolution theorem):

$$\|f * g\|_{pq}^{*} \leqq B \|f\|_{p_0 q_0}^{*} \|g\|_{p_1 q_1}^{*},$$

where $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1 < p_0, p_1 < \infty$ *and* $0 \leqq 1/q = 1/q_0 + 1/q_1 \leqq 1$.

Section 5. REFERENCES

Various properties of $L(p, q)$ spaces have appeared in many places, often as special cases of a more general theory. We will mention several places where related results and applications are found. The references given are not necessarily the first or the only place where the indicated result appears.

The principal references are [19] and [20], where G. G. Lorentz defines special cases of $L(p, q)$ spaces and proves many of their properties. The notion of a non-increasing rearrangement of a function was used by Hardy Littlewood and Payley. (See [32].) A simple proof of the inequality $\|f\|_{pq_2}^{*} \leqq B \|f\|_{pq_1}^{*}$, $q_1 \leqq q_2$, is found in O'Neil [22]. The technique used in the proof