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Autor(en): **Meir, A.**

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A NEW FAMILY OF LINEAR TRANSFORMATIONS

by A. MEIR

1. In 1935 J. Karamata [6] defined a linear transformation method which in 1953 was rediscovered by Lototsky [8] and was published in the survey article of R. P. Agnew [2]. Let $\{P_{nk}\}$ ($k=0, 1, \dots, n; n=0, 1, \dots$) be defined by the identity

$$(1.1) \quad x(x+1) \dots (x+n-1) = \sum_{k=0}^n P_{nk} x^k,$$

and let $\{t_n\}_0^\infty$, the Lototsky transform of a sequence $\{s_n\}_0^\infty$ be

$$t_n = \frac{1}{n!} \sum_{k=0}^n P_{nk} s_k.$$

Some immediate generalizations were given by Wuckovic [11] who replaced the left hand side of (1.1) by $(x+\alpha) \dots (x+\alpha+n-1)$ ($\alpha > -1$) and by Harlestad [4] who replaced it by $(x+q-1) \dots (x+nq-1)$, ($q > 0$). Jakimovski's result [5] included all the previous ones as special cases by defining the $[F, d_n]$ transformation for any fixed sequence of real or complex numbers $\{d_n\}_1^\infty$ ($d_n \neq -1$) by

$$(x+d_1)(x+d_2) \dots (x+d_n) = \sum_{k=0}^n P_{nk} x^k.$$

Regularity, inclusion, shift-properties and applications to analytic continuation, Fourier series, etc., have been thoroughly investigated (see [9], [7]).

Here we give certain further possible generalizations of the above methods, some of their main properties and their relations to well-known classical summation methods.

2. For given sequences of real or complex numbers $\{c_n\}_1^\infty$, $\{d_n\}_1^\infty$ ($c_n \neq -1$, $d_n \neq -1$) let $C_0(x) = D_0(x) = 1$ and for $n \geq 1$

$$(2.1) \quad \begin{cases} C_n(x) = (x+c_1)(x+c_2) \dots (x+c_n) \\ D_n(x) = (x+d_1)(x+d_2) \dots (x+d_n) \end{cases}$$

and let the matrix $\{r_{nk}\}$ ($n=0, 1, \dots; k=0, \dots, n$) be defined by $r_{00} = 1$

and for $n \geq 1$ by

$$(2.2) \quad \sum_{k=0}^n r_{nk} C_k(x) / C_k(1) = D_n(x) / D_n(1).$$

A sequence $\{s_n\}_0^\infty$ will be called $[F, c, d]$ -summable to σ if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n r_{nk} s_k = \sigma$$

The $[F, d_n]$ -transformation is the special case $c_n = 0$ for $n = 1, 2, \dots$. Other special examples are the following: (i) If $d_n = 0$ ($n=1, 2, \dots$) and for some fixed $q > 0$, $c_n = q^{n-1}$ ($n=1, 2, \dots$) then (2.2) yields

$$r_{nk} = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=1}^n (1 + q^{j-1})$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the q -binomial coefficients (see e.g. [1]). (ii) If $d_n = n$, $c_n = q^{n-1}$ ($n=1, 2, \dots$) then (2.2) is equivalent to formula (2) of Carlitz [3]. (iii) If $d_n = 0$ ($n=1, 2, \dots$) then $\{r_{nk}\}$ is the inverse of $[F, c_n]$ -transformation (see Formula (8.7) in [9]). (iv) If $c_k = d_{k-1}$ ($k=2, 3, \dots$), $c_1 = 0$ then $\{r_{nk}\}$ is the matrix transforming the $[F, d_n]$ -sum of $0, s_0, s_1, \dots$ into the $[F, d_n]$ -sum of s_0, s_1, \dots .

3. Regularity. The following recursion relation is an easy conclusion from (2.2)

$$(3.1) \quad (1 + d_n) r_{nk} = (d_n - c_{k+1}) r_{n-1,k} + (1 + c_k) r_{n-1,k-1}$$

where $r_{00} = 1$ and $r_{nk} = 0$ if $k > n$, if $k < 0$ or if $n < 0$. Iterating (3.1) one shows by induction that for $n \geq k \geq 1$ we have

$$(3.2) \quad r_{nk} = \sum_{v=k}^n b_{nv} r_{v-1,k-1}$$

where for $k \leq v \leq n$

$$(3.3) \quad b_{nv} \equiv b_{nv}^{(k)} = \frac{1 + c_k}{1 + d_v} \prod_{j=v+1}^n \left(1 - \frac{1 + c_{k+1}}{1 + d_j} \right)$$

(an empty product being 1 by definition). We formulate now the following

THEOREM. *Suppose $\{c_n\}_1^\infty, \{d_n\}_1^\infty$ satisfy*

$$(3.4) \quad 0 < (1 + c_k)(1 + d_{k+v}) \leq 1, \quad k = 1, 2, \dots; v = 0, 1, \dots$$

$$(3.5) \quad \sum_{n=1}^{\infty} |1 + d_n|^{-1} = +\infty.$$

Then $[F, c, d]$ -method of summation is regular.

Proof. First we observe that from (3.4) it follows by (3.1) that $r_{nk} \geq 0$ for all n and k and from (2.2) that $\sum_{k=0}^n r_{nk} = 1$ for all n . Thus to prove regularity it remains to show that for fixed $k = 0, 1, \dots$ we have $r_{nk} = 0$ (1) as $n \rightarrow \infty$. Now, from (3.1) $r_{n0} = (d_n - c_1)(1 + d_n)^{-1} r_{n-1,0}$, since $r_{00} = 1$, we have

$$r_{n0} = \prod_{k=1}^n \left(1 - \frac{1 + c_1}{1 + d_k} \right)$$

and by (3.4)

$$\leq \exp \left\{ - |1 + c_1| \sum_{k=1}^n |1 + d_k|^{-1} \right\}.$$

By (3.5) then

$$(3.6) \quad \lim_{n \rightarrow \infty} r_{n0} = 0.$$

We assume now that for some $k \geq 1$

$$\lim_{n \rightarrow \infty} r_{n,k-1} = 0$$

and prove that then $\lim_{n \rightarrow \infty} r_{nk} = 0$ too. This will follow from (3.2) if we establish that, for fixed k , $\{b_{nv}\}$ satisfy the inequalities

$$(3.7) \quad \sum_{v=k}^n |b_{nv}| \leq H < \infty, \quad n = k, k + 1, \dots$$

$$(3.8) \quad \lim_{n \rightarrow \infty} b_{nv} = 0, \quad v = k, k + 1, \dots$$

Now from (3.3), $b_{nv} \geq 0$ for all n, v and by an easy calculation

$$\begin{aligned} \sum_{v=k}^n b_{nv} &= \frac{1 + c_k}{1 + c_{k+1}} \left\{ 1 - \frac{d_k - c_{k+1}}{1 + d_k} \prod_{j=k+1}^n \left(1 - \frac{1 + c_{k+1}}{1 + d_j} \right) \right\} \\ &\leq \left| \frac{1 + c_k}{1 + c_{k+1}} \right| \left\{ 1 + \left| \frac{d_k - c_{k+1}}{1 + d_k} \right| \right\} \end{aligned}$$

which proves (3.7). Also by (3.4) for fixed $v \geq k$,

$$|b_{nv}| \leq \frac{1 + c_k}{1 + d_v} \exp \left\{ - |1 + c_{k+1}| \sum_{j=v+1}^n |1 + d_j|^{-1} \right\}$$

which by (3.5)

$$\leq 0(1), \quad \text{as } n \rightarrow \infty.$$

This proves (3.8) and thus by induction from (3.6)

$$\lim_{n \rightarrow \infty} r_{nk} = 0, \quad k = 0, 1, \dots$$

completing the proof of the Theorem.

4. The $\{r_{nk}\}$ defined by (2.2) or (3.1) can be expressed on using divided differences (see e.g. [10] p. 8, formula 1.4 (1)) as follows

$$(4.1) \quad r_{nk} = (1 + c_1)(1 + c_2) \dots (1 + c_k) [-c_1, -c_2, \dots, -c_{k+1}; D_n]$$

where $[x_1, x_2, \dots, x_k; f]$ denotes the usual divided differences of a function $f(x)$ at x_1, x_2, \dots, x_k . This formula suggests another family of possible transformation methods. For a function $f(x)$ with $f(0) = 1$, for a given sequence $\{c_k\}_1^\infty$ and for $\lambda > 0$, let for $k \geq 0$

$$(4.2) \quad r_k(\lambda) = (-1)^k (\lambda + c_1) \dots (\lambda + c_k) [\lambda + c_1, \dots, \lambda + c_{k+1}; f].$$

If for a sequence $\{s_n\}_0^\infty$ and for $\lambda \geq \lambda_0$

$$t(\lambda) = \sum_{k=0}^{\infty} r_k(\lambda) s_k$$

exists and $\lim_{\lambda \rightarrow \infty} t(\lambda) = \sigma$, then we say the sequence $\{s_n\}$ is $\{f, c\}$ summable

to σ . In particular if $f(x) = (1+x)^{-1}$ then

$$r_k(\lambda) = \frac{(\lambda + c_1) \dots (\lambda + c_k)}{(1 + \lambda + c_1) \dots (1 + \lambda + c_{k+1})}$$

and this transformation method reduces to the classical Abel method if $c_k = 0$ ($k=1, 2, \dots$).

Another modification of (4.1) would suggest to define $\{r_{nk}\}$ for a function $f(x)$ and a given sequence $\{c_n\}$ by

$$r_{nk} = (-1)^{n-k} c_{k+1} c_{k+2} \dots c_n [c_k, \dots, c_n; f]$$

which, if $f(c_n) = \mu_n$, generates the well known generalized Hausdorff $[H, c_n; \mu_n]$ -transformation methods.

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A. Meir
Department of Mathematics
The University of Alberta
Edmonton, Canada

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