NEW FAMILY OF LINEAR TRANSFORMATIONS

A NEW FAMILY OF LINEAR TRANSFORMATIONS

by A. MEIR

1. In 1935 J. Karamata [6] defined a linear transformation method which in 1953 was rediscovered by Lototsky [8] and was published in the survey article of R. P. Agnew [2]. Let $\{ P_{nk} \}$ $(k=0, 1, \ldots, n; n=0, 1, \ldots)$ be defined by the identity

$$(1.1) \qquad x(x+1)\ldots(x+n-1) = \sum_{k=0}^{n} P_{nk}\, x^k ,$$

and let $\{ t_n \}_0^\infty$, the Lototsky transform of a sequence $\{ s_n \}_0^\infty$ be

$$t_n = \frac{1}{n\,!} \sum_{k=0}^{n} P_{nk}\, s_k .$$

Some immediate generalizations were given by Wuckovic [11] who replaced the left hand side of (1.1) by $(x+\alpha)\ldots(x+\alpha+n-1)$ $(\alpha > -1)$ and by Harlestam [4] who replaced it by $(x+q-1)\ldots(x+nq-1)$, $(q>0)$. Jakimovski's result [5] included all the previous ones as special cases by defining the $[F, d_n]$ transformation for any fixed sequence of real or complex numbers $\{ d_n \}_1^\infty$ $(d_n \neq -1)$ by

$$(x+d_1)(x+d_2)\ldots(x+d_n) = \sum_{k=0}^{n} P_{nk}\, x^k .$$

Regularity, inclusion, shift-properties and applications to analytic continuation, Fourier series, etc., have been thoroughly investigated (see [9], [7]).

Here we give certain further possible generalizations of the above methods, some of their main properties and their relations to well-known classical summation methods.

2. For given sequences of real or complex numbers $\{ c_n \}_1^\infty$, $\{ d_n \}_1^\infty$ $(c_n \neq -1, d_n \neq -1)$ let $C_0(x) = D_0(x) = 1$ and for $n \geqq 1$

$$(2.1) \qquad \begin{cases} C_n(x) = (x+c_1)(x+c_2)\ldots(x+c_n) \\ D_n(x) = (x+d_1)(x+d_2)\ldots(x+d_n) \end{cases}$$

and let the matrix $\{ r_{nk} \}$ $(n=0, 1, \ldots; k=0, \ldots, n)$ be defined by $r_{00} = 1$

and for $n \geqq 1$ by

$$\text{(2.2)} \qquad \sum_{k=0}^{n} r_{nk}\, C_k(x) / C_k(1) = D_n(x) / D_n(1)\,.$$

A sequence $\{ s_n \}_0^\infty$ will be called $[F, c, d]$-summable to σ if

$$\lim_{n \to \infty} \sum_{k=0}^{n} r_{nk}\, s_k = \sigma$$

The $[F, d_n]$-transformation is the special case $c_n = 0$ for $n = 1, 2, \ldots$. Other special examples are the following: (i) If $d_n = 0$ $(n=1, 2, \ldots)$ and for some fixed $q > 0$, $c_n = q^{n-1}$ $(n=1, 2, \ldots)$ then (2.2) yields

$$r_{nk} = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=1}^{n} (1 + q^{j-1})$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ denotes the q-binomial coefficients (see e.g. [1]). (ii) If $d_n = n$, $c_n = q^{n-1}$ $(n=1, 2, \ldots)$ then (2.2) is equivalent to formula (2) of Carlitz [3]. (iii) If $d_n = 0$ $(n=1, 2, \ldots)$ then $\{ r_{nk} \}$ is the inverse of $[F, c_n]$-transformation (see Formula (8.7) in [9]). (iv) If $c_k = d_{k-1}$ $(k=2, 3, \ldots)$, $c_1 = 0$ then $\{ r_{nk} \}$ is the matrix transforming the $[F, d_n]$-sum of $0, s_0, s_1, \ldots$ into the $[F, d_n]$-sum of $s_0, s_1, \ldots$.

3. Regularity. The following recursion relation is an easy conclusion from (2.2)

$$\text{(3.1)} \qquad (1 + d_n)\, r_{nk} = (d_n - c_{k+1})\, r_{n-1,k} + (1 + c_k)\, r_{n-1,k-1}$$

where $r_{00} = 1$ and $r_{nk} = 0$ if $k > n$, if $k < 0$ or if $n < 0$. Iterating (3.1) one shows by induction that for $n \geqq k \geqq 1$ we have

$$\text{(3.2)} \qquad r_{nk} = \sum_{\nu=k}^{n} b_{n\nu}\, r_{\nu-1,k-1}$$

where for $k \leqq \nu \leqq n$

$$\text{(3.3)} \qquad b_{n\nu} \equiv b_{n\nu}^{(k)} = \frac{1 + c_k}{1 + d_\nu} \prod_{j=\nu+1}^{n} \left(1 - \frac{1 + c_{k+1}}{1 + d_j} \right)$$

(an empty product being 1 by definition). We formulate now the following

THEOREM. *Suppose* $\{ c_n \}_1^\infty$, $\{ d_n \}_1^\infty$ *satisfy*

$$\text{(3.4)} \qquad 0 < (1 + c_k)(1 + d_{k+\nu}) \leqq 1\,, \qquad k = 1, 2, \ldots;\ \nu = 0, 1, \ldots$$

(3.5) $$\sum_{n=1}^{\infty} |1+d_n|^{-1} = +\infty .$$

Then [F, c, d]-method of summation is regular.

Proof. First we observe that from (3.4) it follows by (3.1) that $r_{nk} \geqq 0$ for all n and k and from (2.2) that $\sum_{k=0}^{n} r_{nk} = 1$ for all n. Thus to prove regularity it remains to show that for fixed $k = 0, 1, \ldots$ we have $r_{nk} = 0\,(1)$ as $n \to \infty$. Now, from (3.1) $r_{n0} = (d_n - c_1)(1+d_n)^{-1} r_{n-1,0}$, since $r_{00} = 1$, we have

$$r_{n0} = \prod_{k=1}^{n} \left(1 - \frac{1+c_1}{1+d_k}\right)$$

and by (3.4)

$$\leqq \exp\left\{-|1+c_1| \sum_{k=1}^{n} |1+d_k|^{-1}\right\}.$$

By (3.5) then

(3.6) $$\lim_{n\to\infty} r_{n0} = 0 .$$

We assume now that for some $k \geqq 1$

$$\lim_{n\to\infty} r_{n,k-1} = 0$$

and prove that then $\lim_{n\to\infty} r_{nk} = 0$ too. This will follow from (3.2) if we establish that, for fixed k, $\{b_{n\nu}\}$ satisfy the inequalities

(3.7) $$\sum_{\nu=k}^{n} |b_{n\nu}| \leqq H < \infty , \qquad n = k, k+1, \ldots$$

(3.8) $$\lim_{n\to\infty} b_{n\nu} = 0 , \qquad \nu = k, k+1, \ldots .$$

Now from (3.3), $b_{n\nu} \geqq 0$ for all n, ν and by an easy calculation

$$\sum_{\nu=k}^{n} b_{n\nu} = \frac{1+c_k}{1+c_{k+1}} \left\{1 - \frac{d_k - c_{k+1}}{1+d_k} \prod_{j=k+1}^{n} \left(1 - \frac{1+c_{k+1}}{1+d_j}\right)\right\}$$

$$\leqq \left|\frac{1+c_k}{1+c_{k+1}}\right| \left\{1 + \left|\frac{d_k - c_{k+1}}{1+d_k}\right|\right\}$$

which proves (3.7). Also by (3.4) for fixed $\nu \geqq k$,

$$| b_{n\nu} | \leqq \frac{1 + c_k}{1 + d_\nu} \exp \left\{ - | 1 + c_{k+1} | \sum_{j=v+1}^{n} | 1 + d_j |^{-1} \right\}$$

which by (3.5)

$$\leqq 0(1) , \quad \text{as} \quad n \to \infty .$$

This proves (3.8) and thus by induction from (3.6)

$$\lim_{n \to \infty} r_{nk} = 0 , \quad k = 0, 1, \ldots$$

completing the proof of the Theorem.

4. The $\{ r_{nk} \}$ defined by (2.2) or (3.1) can be expressed on using divided differences (see e.g. [10] p. 8, formula 1.4 (1)) as follows

(4.1) $$r_{nk} = (1+c_1)(1+c_2) \ldots (1+c_k) [-c_1, -c_2, \ldots, -c_{k+1}; D_n]$$

where $[x_1, x_2, \ldots, x_k; f]$ denotes the usual divided differences of a function $f(x)$ at $x_1, x_2, \ldots, x_k$. This formula suggests another family of possible transformation methods. For a function $f(x)$ with $f(0) = 1$, for a given sequence $\{ c_k \}_1^\infty$ and for $\lambda > 0$, let for $k \geqq 0$

(4.2) $$r_k(\lambda) = (-1)^k (\lambda + c_1) \ldots (\lambda + c_k) [\lambda + c_1, \ldots, \lambda + c_{k+1}; f] .$$

If for a sequence $\{ s_n \}_0^\infty$ and for $\lambda \geqq \lambda_0$

$$t(\lambda) = \sum_{k=0}^{\infty} r_k(\lambda) s_k$$

exists and $\lim_{\lambda \to \infty} t(\lambda) = \sigma$, then we say the sequence $\{ s_n \}$ is $\{ f, c \}$ summable to σ. In particular if $f(x) = (1+x)^{-1}$ then

$$r_k(\lambda) = \frac{(\lambda + c_1) \ldots (\lambda + c_k)}{(1 + \lambda + c_1) \ldots (1 + \lambda + c_{k+1})}$$

and this transformation method reduces to the classical Abel method if $c_k = 0$ $(k=1, 2, \ldots)$.

Another modification of (4.1) would suggest to define $\{ r_{nk} \}$ for a function $f(x)$ and a given sequence $\{ c_n \}$ by

$$r_{nk} = (-1)^{n-k} c_{k+1} c_{k+2} \ldots c_n [c_k, \ldots, c_n; f]$$

which, if $f(c_n) = \mu_n$, generates the well known generalized Hausdorff $[H, c_n; \mu_n]$-transformation methods.

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