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A CONDITION FOR EXISTENCE
OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN
COLLECTION OF SETS

by Arthur B. BROWN and Gerald FREILICH

The origin of this note lies in an oversight appearing in [1] and [2], a difficulty that was already realized by the translators of [1]. (See Translators' Note in [1], page 16. Since situations arise in which B -algebras with different units are used, Theorem 4 on page 19 of [1] and on page 25 of [2] requires an additional hypothesis. See the theorem below.) It is hoped that the present note will be of independent interest.

DEFINITIONS. *A σ -ring (of sets) is a non-empty collection of sets closed under the operations of difference (of a pair of sets) and countable union.*

A Borel algebra, or B -algebra, is a σ -ring which has an element that contains every other element of the σ -ring. The (unique) maximal element is called the unit of the B -algebra.

A member of a collection of sets is called the smallest member if it is contained in every other member of the collection.

LEMMA. *If S is a non-empty collection of sets each contained in a set X , then there exists a smallest B -algebra $B(S)$ with unit X containing S .*

Proof. Take $B(S)$ to be the intersection of all B -algebras with unit X that contain S .

If we want now to generalize the lemma by omitting the requirement that the B -algebras under consideration have the same unit X , an additional hypothesis is necessary.

THEOREM. *Let S be a non-empty collection of sets whose union is X . Then there is a smallest B -algebra containing S if and only if X is the union of some countable collection of sets of S . If there is a smallest B -algebra containing S , then that algebra has unit X and is the algebra $B(S)$ of the Lemma.*

Proof. Suppose that X is the union of a countable collection of sets of S . Let W be any B -algebra containing S , where sets of W are not restricted to be subsets of X , and let $D = W \cap B(S)$, where $B(S)$ is the smallest

B -algebra with unit X and containing S . (See Lemma.) Since W is a σ -ring, $X \in W$; hence $X \in D$. Since $D \subseteq B(S)$, the sets in D are subsets of X . Since W and $B(S)$ are σ -rings, so is D . Hence D is a B -algebra with unit X . Since $B(S)$ is the smallest B -algebra with unit X , we infer that $B(S) \subseteq D$, and hence $B(S) = D$. Thus $B(S) = D \subseteq W$, so $B(S)$ is the smallest B -algebra containing S , as was to be proved.

Now suppose X is not the union of any countable collection of sets of S . Choose $\alpha \notin X$ and let $Y = X \cup \{\alpha\}$. Let $S' = \{A: A \subseteq \cup_{i=1}^{\infty} S_i, S_i \in S\}$, $S'' = \{A: A \in S' \text{ or } (Y-A) \in S'\}$. Then any subset of a member of S' is a member of S' , and S' is clearly a σ -ring. It is obvious that S'' contains S . We now prove that S'' is a B -algebra.

Since $\emptyset \in S'$, $Y \in S''$, so Y will be the unit. Let A_1, A_2, \dots be members of S'' . If each $A_j \in S'$, with $A_j \subseteq \cup_{i=1}^{\infty} S_{ij}$, $S_{ij} \in S$, then $\cup_{j=1}^{\infty} A_j \subseteq \cup_{j=1}^{\infty} \cup_{i=1}^{\infty} S_{ij}$, so $\cup_{j=1}^{\infty} A_j \in S' \subseteq S''$. If some $A_k \notin S'$, then $Y - A_k \in S'$. Hence $Y - \cup_{j=1}^{\infty} A_j = \cap_{j=1}^{\infty} (Y - A_j) \subseteq Y - A_k \in S'$, so that $Y - \cup_{j=1}^{\infty} A_j \in S'$ and consequently $\cup_{j=1}^{\infty} A_j \in S''$. Thus it is proved that S'' is closed under countable unions. We now consider differences.

Suppose $\{A, B\} \subseteq S''$. If $A \in S'$ then $A - B \subseteq A \in S'$, so $A - B \in S'$ and hence $A - B \in S''$. If $A \notin S'$ then $Y - A \in S'$, and if $B \in S'$ we have $Y - (A - B) \subseteq (Y - A) \cup B \in S'$, so $Y - (A - B) \in S'$; hence $A - B \in S''$. If $A \notin S'$ and $B \notin S'$, then $A - B = (Y - B) - (Y - A) \in S' \subseteq S''$. This completes the proof that S'' is a B -algebra.

Since X is not the union of any countable collection of sets of S , it is clear that $X \notin S'$. Consequently $X \notin S''$, for if $X = Y - A$ with $A \in S'$, we would have $\alpha \in X$, contrary to the choice of α . We are now in a position to complete the proof.

If there were a smallest B -algebra V containing S , then by the definition of unit, the unit E of V would contain X . Furthermore, V would be contained in the set of all subsets of X (the latter being a B -algebra containing S), so $E \subseteq X$. Hence X would be the unit E of V . Then, from $X \in V$ and $X \notin S''$, it would follow that $V \not\subseteq S''$, contrary to the fact that S'' is a B -algebra containing S . Hence there is no smallest B -algebra containing S .

EXAMPLES. Let X be an uncountable set and let S be the set of all countable subsets of X . By the Theorem, there is no smallest B -algebra containing S .

Note that the Theorem implies that if S is a non-empty collection of sets such that there is no smallest B -algebra containing S , then the union of the members of S must be uncountable.

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