Definitions

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4) Let P_1 and P_2 be any two distinct points on $\bar{E}$, $Q_1 = f(P_1)$, and $Q_2 = f(P_2)$. Let $P_1 P_2$ denote the closed interval determined by P_1 and P_2 and $Q_1 Q_2$ the closed interval determined by Q_1 and Q_2. Let the curve $C = f(P_1 P_2)$. Then there exists a point R on C such that the tangent line to C at R is parallel to $Q_1 Q_2$.

5) With the notation as in 4), let the *deviation* $D(P_1 P_2)$ denote the L.U.B. of the acute angles φ between the surface chord $Q_1 Q_2$ and any tangent line to C. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < \rho(P_1, P_2) < \delta$ then $D(P_1 P_2) < \epsilon$.

6) For every $\epsilon > 0$ there exists $\delta > 0$ such that if P_1 and P_2 are any two distinct points of $\bar{E}$ such that $\rho(P_1, P_2) < \delta$ then $\psi < \epsilon$, where ψ is the acute angle between the surface normals at $f(P_1)$ and at $f(P_2)$.

We need to give some preliminary definitions.

Definitions

We shall call a surface $S = f(\bar{E})$ simple when the boundary of $\bar{E}$ is a simple closed polygon. We shall first be concerned only with simple surfaces.

A polyhedron Π is said to be inscribed on S when all the vertices of Π are in S and the orthogonal projection, Proj Π, on the xy plane is $\bar{E}$. By the norm of a polyhedron we shall mean the greatest of the diameters of the faces (triangles) of Π.

Let Π be inscribed on S and let A be a face of Π. By the deviation $D(A)$ of A we shall mean the L.U.B. of the acute angles between the normal to A and the surface normal at a point of the surface *subtended* by A. By the deviation norm of Π we shall mean the greatest of the deviations of its faces.

We shall consider sequences of polyhedra which are inscribed on S. A sequence $\{\Pi_1, \Pi_2, \dots\}$ of such polyhedra is said to be a *proper* sequence of polyhedra inscribed on S when the corresponding sequence of norms $\{N_1, N_2, \dots\}$ converges to zero and the corresponding sequence $\{\phi_1, \phi_2, \dots\}$ of deviation norms also converges to zero.

We now give our basic definition of surface area:

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. If to every proper sequence of polyhedra inscribed on $S = f(\bar{E})$ the corresponding sequence of polyhedral areas $\{A_1, A_2, \dots\}$ converges, then

then we say that S is *quadrable* anf that the necessarily unique limit of $\{A_1, A_2, \ldots\}$ is the area of the surface S.

THEOREM 1.

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then there exist a proper sequence $\{\Pi_1, \Pi_2, \ldots\}$ of polyhedra inscribed on S.

Proof:

For every positive number r there exists a decomposition of $\bar{E}$ as the union of closed right triangles whose diameters are all less than r. The vertices of these right triangles determine a finite set of points in S whose projection is precisely the set of these vertices. This set of points in S determines a triangular polyhedron which is inscribed on S. We shall show that by making the norm of the decomposition of $\bar{E}$ sufficiently small we can make the acute angle between the normal to each polyhedral face and the surface normal at any point of the portion of S which is subtended by the particular face to be arbitrarily small. Let $\varepsilon > 0$ be given.

By property 3) there exist positive real numbers $k < 1$ and δ_1 such that if PP_1P_2 is a right triangle on $\bar{E}$ (P being the right angled vertex) with diameter $< \delta_1$, then $|\cos(\overrightarrow{QQ_1}, \overrightarrow{QQ_2})| < k$. Let the decomposition of $\bar{E}$ by right triangles be of norm less than δ_1.

By property 1) there exists a positive real number θ such that if $|\sin(\overrightarrow{QQ_1}, \overrightarrow{QQ_1'})| < \theta$ and $|\sin(\overrightarrow{QQ_2}, \overrightarrow{QQ_2'})| < \theta$, then the acute angle between $\overrightarrow{QQ_1} \times \overrightarrow{QQ_2}$ and $\overrightarrow{QQ_1'} \times \overrightarrow{QQ_2'}$ is less than $\varepsilon / 3$.

By properties 4) and 5) there exists a positive real number δ_2 such that if PP_1P_2 is a right triangle on $\bar{E}$ with diameter less than δ_2, then the angle between the chord $\overrightarrow{QQ_1}$ and the tangent line at Q to the curve on S subtented by $\overrightarrow{QQ_1}$ is less than θ. Similarly, the angle between the chord $\overrightarrow{QQ_2}$ and the tangent line at Q to the curve on S subtended by $\overrightarrow{QQ_2}$ is less than θ. It follows that the angle between the normal to the polyhedral face $QQ_1\,Q_2$ and the surface normal at Q is less than $\varepsilon / 3$.

By property 6) there exists a positive real number δ_3 such that if the diameter of the triangle $PP_1\,P_2$ is less than δ_3, then the angle between the surface normals at any two points of the portion of S which is subtended by the polyhedral face $QQ_1\,Q_2$ is less than $\varepsilon / 3$.

Let δ be the least of δ_1, δ_2, and δ_3. If D is any decomposition of $\bar{E}$ int closed right triangles of norm less than δ, then if $QQ_1\,Q_2$ is any of the polyhedral faces, the L.U.B. of the angles between the normal to $QQ_1\, Q_2$ and the surface normals at any point of the portion of the surface subtended by $QQ_1\, Q_2$ is less than ε.

Thus corresponding to a sequence $\{\varepsilon_1, \varepsilon_2, \ldots\}$ converging to zero, there exists a sequence of polyhedra with corresponding sequence of norms converging to zero and also with corresponding sequence of deviation norms converging to zero.

THEOREM 2.

Let E be an open set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then for every proper sequence of polyhedra inscribed on S the corresponding sequence $\{A_1, A_2, \ldots\}$ of polyhedral areas converges and moreover it converges to the double integral

$$\int\limits_{\bar{E}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\; d(x, y)\,.$$

Proof:

For each n, the projection of the faces of Π_n constitute a decomposition D_n of $\bar{E}$ as the union of a finite set of closed triangles. Let the triangle $\Delta_{mn} = QQ_1\, Q_2$ be the face of Π_n and let $\Delta'_{mn} = \text{Proj}\; QQ_1\, Q_2 = PP_1\, P_2$. Let β_{mn} be the acute angle between the normals to Δ_{mn} and to Δ'_{mn}. Let A_{mn} and A'_{mn} denote the areas of Δ_{mn} and Δ'_{mn}, respectively. Then $A_{mn} = A'_{mn}$ sec β_{mn} and the area A_n of Π_n is $\sum\limits_m A'_{mn}$ sec β_{mn}.

Let P_{mn} be any point in Δ'_{mn} and let Q_{mn} be the point of S whose projection is P_{mn}. Let θ_{mn} denote the acute angle between the surface normal at Q_{mn} and the z-axis.

Let $\{\Pi_1, \Pi_2, \Pi_3, ..\}$ be any proper sequence of polyhedra inscribed on S. We shall associate to $\{\Pi_1, \Pi_2, \Pi_3, \ldots\}$ certain related sequences.

$$\Pi_1,\ \Pi_2,\ \Pi_3,\ \ldots$$

$$\phi_1,\ \phi_2,\ \phi_3,\ \ldots$$

$$\Sigma_1,\ \Sigma_2,\ \Sigma_3,\ \ldots$$

$$\Sigma'_1,\ \Sigma'_2,\ \Sigma'_3,\ \ldots$$

The sequence $\{\phi_1, \phi_2, \phi_3, \ldots\}$ is the corresponding sequence of deviation norms. The sequence $\{\Sigma_1, \Sigma_2, \Sigma_3, ..\}$ is the corresponding sequence of polyhedral areas. $\Sigma_n = \underset{m}{\Sigma} A'_{mn} \sec \beta_{mn}$. In the fourth sequence $\Sigma'_n = \underset{m}{\Sigma} A'_{mn}$ $\sec \theta_{mn}$. Here $\sec \theta_{mn}$ is the value of $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ at some point of A'_{mn}. Thus the sequence $\{\Sigma'_1, \Sigma'_2, \Sigma'_3, \ldots\}$ is a sequence of Riemann sums of the function $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ on $\bar{E}$ with corresponding sequence of norms converging to zero. Since $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ is continuous on $\bar{E}$, this converges to the double integral $\oint = \int_{\bar{E}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, d(x, y)$.

We will now consider the sequence $\{\Sigma_1, \Sigma_2, \Sigma_3, \ldots\}$.

Let θ denote the acute angle between the surface normal at a point of S and the z-axis. Sec $\theta = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ is bounded on $\bar{E}$. Thus there exists an acute angle $\theta^* > 0$ such that $\theta < \theta^*$ for all points of $\bar{E}$ (i.e. for all points of S). Since $\sec \theta$ is uniformly continuous on the closed interval $[0, \theta^*]$, for every $\eta > 0$ there exists $\tau > 0$ such that if $0 < \theta_1 < \theta^*, 0 < \theta_2 < \theta^*$, and $|\theta_1 - \theta_2| < \tau$, then $|\sec \theta_1 - \sec \theta_2| < \eta$.

We now compare the corresponding sequences

$$\{\Sigma_1, \Sigma_2, \Sigma_3, \ldots\}$$
$$\{\Sigma'_1, \Sigma'_2, \Sigma'_3, \ldots\}.$$

Let $\varepsilon > 0$ be given. Take $\frac{\varepsilon}{2A}$, where A = area of $\bar{E}$. There exists $\tau > 0$ such that if $|\theta_1 - \theta_2| < \tau$, then $|\sec \theta_1 - \sec \theta_2| < \frac{\varepsilon}{2A}$. Since $\{\phi_1, \phi_2, \phi_3, \ldots\}$ converges to zero, there exists a positive integer N_1 such that if $n > N_1$ then $\phi_n < \tau$. Thus if $n > N_1$, then

$$|\Sigma_n - \Sigma'_n| = |\underset{m}{\Sigma} A'_{mn} (\sec \beta_{mn} - \sec \theta_{mn})| < \frac{\varepsilon}{2A} \underset{m}{\Sigma} A'_{mn} = \frac{\varepsilon}{2}.$$

Since $\{\Sigma'_n, \Sigma'_2, \Sigma'_3, \ldots\}$ converges to $\oint$, there exists a positive integer N_2. such that if $n > N_2$, then $|\Sigma'_n - \oint| < \frac{\varepsilon}{2}$. Let N be the larger of N_1 and N_2. If $n > N$ then

$$|\Sigma_n - \oint| = |\Sigma_n - \Sigma'_n + \Sigma'_n - \oint| \leqslant |\Sigma_n - \Sigma'_n| + |\Sigma'_n - \oint| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon.$$

Thus $\{\Sigma_1, \Sigma_2, \Sigma_3, \ldots\}$ converges to $\oint$.

Thus far we have defined the concept of area only for surfaces which are not only continuously differentiable but are also simple. We now remove this latter restriction.

Let E be any quadrable (i.e. Jordan measurable) open set on the xy plane having for boundary a simple closed curve. Let f be defined and continuously differentiable on $\bar{E}$. Let P be any subset of $\bar{E}$ whose boundary is a simple closed polygon. The surface $S_p = f(P)$ is quadrable. Denote its area by A_p. Consider now the set of all such areas A_p. Since sec θ is bounded on $\bar{E}$, for every polygonal subset P of $\bar{E}$, $A_p \leqq AM$, where A is the area of $\bar{E}$ and M is an upper bound of $|\sec \theta|$ on $\bar{E}$. We now define the area of $S = f(\bar{E})$ as the L.U.B. of the set [all A_p].

THEOREM 3.

Let E be a quadrable open set on the xy plane having for boundary a simple closed curve. Let f be defined and continuously differentiable on $\bar{E}$. Then the area of $S = f(\bar{E})$ is given by

$$\oint = \int_{\bar{E}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, d(x, y)\,.$$

Proof:

Let B denote the L.U.B. of the set [all A_p]. For each P, $A_p \leqslant \oint$ and hence $B \leqslant \oint$. Suppose now that $\oint - B = 2\varepsilon > 0$.

Let $\{D_1, D_2, D_3, \ldots\}$ be any sequence of triangular " decompositions " of $\bar{E}$ with corresponding sequence of norms converging to zero. Here we permit the triangles to abut beyond the boundary of $\bar{E}$. On each D_n form a Riemann sum of $F(x, y) = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ in the following manner: If a triangle does not abut beyond the boundary of $\bar{E}$, then take for the point P any point of the triangle. However, if a triangle does abut beyond the boundary of $\bar{E}$, let its contribution to the Riemann sum be zero. Now every sequence $\{S_1, S_2, S_3, \ldots\}$ of such Riemann sums converges and moreover, it converges to $\oint$. Since $\{S_1, S_2, S_3, \ldots\}$ converges to $\oint$, there exists a positive integer N such that if $n > N$ then $|\oint - S_n| < \frac{\varepsilon}{2}$.

On D_n, the set of the triangles which do not abut beyond the boundaries of $\bar{E}$ constitutes a polygonal subset of $\bar{E}$. Call it P_n. There exists a triangular decomposition D_n' of P_n such that if S_n' is a Riemann sum of $f(x, y)$ on D_n', then $|A_{P_n} - S_n'| < \frac{\varepsilon}{4}$ and $|\mathfrak{H} - S_n'| < \frac{\varepsilon}{2}$. It follows that $A_{P_n} > B$. This contradiction shows that $B = \mathfrak{H}$.

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