BOUNDEDNESS THEOREMS FOR SOLUTIONS OF $u''(t) + a(t)f(u)g(u') = 0$ (IV)

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1. Introduction

In our previous work [1-3], we have presented rather fragmentary results concerning the boundedness of solutions to certain second order non-linear differential equations of the following form:

$$u''(t) + a(t)f(u)g(u') = 0 \tag{1.1}$$

where $a(t), f(u)$ and $g(u')$ satisfy certain assumptions to be described below. The purpose of the present paper is to further extend these results and establish comparison theorems. Some of our results presented here may be considered as generalizations to the results of Zhang [4], where a special case of equation (1.1):

$$u''(t) + a(t)f(u) = 0 \tag{1.2}$$

was treated [1]).

Throughout the discussion of this paper, we will need the following assumptions:

(A_1) $g(u')$ is a positive continuous function of u',
(A_2) $f(u)$ is a continuous function of u satisfying $uf(u) > 0$, if $u \neq 0$,
(A_3) $a(t)$ is continuous in t,

$$(A_4)\quad \lim_{|u| \to \infty} \int_0^u f(s)\, ds = \infty,$$

$$(A_5)\quad \lim_{|v| \to \infty} \int_0^v \frac{g(s)}{s\, ds} = \infty.$$

We also list in the following a brief résumé of our previous results on boundedness.

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[1]) Eor other boundedness result concerning (1.2), see [5]-[7], [13], [14].

Theorem (I). Suppose that assumptions A_1, A_2, A_3, and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geqq 0$ for $t \geqq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (I), suppose that assumption A_5 also holds and that $\lim_{t\to\infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (II). Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a'(t) \leqq 0$ for $t \geqq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (II), suppose that assumption A_5 also holds and $\lim_{t\to\infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (III). Suppose that assumptions A_1, A_2, A_3, and A_4 hold and in addition that $a(t) \geqq a_0 > 0$ for $t \geqq T$, and $\int^{\infty} | a'(t) | \, dt < \infty$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (III), suppose that assumption A_5 also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

2. Boundedness Theorems I

Theorem 1. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and that $a(t) > 0$ for $t \geqq T$ and there exists a non-negative function $\alpha(t)$ such that $-a'(t) \leqq \alpha(t)\, a(t)$ with $\int^{\infty} \alpha(s)\, ds < \infty$. Then all solutions of (1.1) are bounded.

Proof. Write equation (1.1) in its system form ($y_1 = u$):

$$\begin{cases} \dfrac{dy_1}{dt} = y_2\,, \\ \\ \dfrac{dy_2}{dt} = -a(t)\,f(y_1)\,g(y_2)\,. \end{cases} \tag{2.1}$$

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_0^{y_1} f(s)\,ds + \frac{1}{a(t)}\int_0^{y_2} \frac{s\,ds}{g(s)} \tag{2.2}$$

Clearly, under the hypothesis of the theorem, we have $V > 0$ whenever $y_1^2 + y_2^2 \neq 0$, and by A_4, $V \to \infty$ as $y_1 \to \infty$. Differentiating with respect to t, we obtain

$$V'(t, y_1, y_2) \leqq -\frac{a'(t)}{a^2(t)}\int_0^{y_2} \frac{s\,ds}{g(s)} \leqq \alpha(t)\,V(t, y_1, y_2)\,,$$

hence,

$$V(t, y_1, y_2) \leqq V(T, y_1(T),\quad y_2(T))\left\{\exp \int_T^t \alpha(s)\,ds\right\} < \infty \tag{2.3}$$

for all t; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption A_5 also holds and $a(t) \leqq a_1$ for all $t \geqq T$, then $u'(t)$ is also bounded for in this case $V \to \infty$ as $y_2 \to \infty$. Thus,

Corollary. In addition to the hypothesis of Theorem 4, suppose that assumption A_5 holds and $a(t) \leqq a_1$ for all $t \geqq T$; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking $\alpha(s) \equiv 0$, Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klokov ([13], Theorem 1). We remark that a slightly weaker version of Klokov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions $u(t)$ and their derivatives $u'(t)$ of the following equation:

$$u''(t) + (1 + e^{-t}\sin t)\,u^{\frac{3}{2}}(t)\left[2 + \cos u(t)\right] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

Theorem 2. Suppose that assumptions A_1, A_2, A_3, and A_4 hold and that $a(t) > 0$, $a(t) \to 0$, and there exists a $\alpha(t) \geqq 0$ such that $a'(t) \leqq \alpha(t)\, a(t)$ while $\int^{\infty} \alpha(s)\, ds < \infty$. Then all solutions of (1.1) satisfy: $a(t)\, F(u(t)) = a(t) \int^{u(t)} f(s)\, ds < \infty$ [1]) for $t \geqq T$, and all its derivatives are bounded.

Proof. Consider the following function:

$$V(t, y_1, y_2) = a(t) \int_0^{y_1} f(s)\, ds + \int_0^{y_2} \frac{s\, ds}{g(s)} \tag{2.4}$$

and note that

$$\begin{aligned} V'(t, y_1, y_2) &\leqq a'(t) \int_0^{y_1} f(s)\, ds \\ &\leqq \alpha(t)\, a(t) \int_0^{y_1} f(s)\, ds \\ &\leqq \alpha(t)\, V(t, y_1, y_2)\, . \end{aligned}$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

Corollary. Under the hypothesis of Theorem 5, if in addition assumption A_5 holds and $a(t) \geqq a_0 > 0$ for $t \geqq T$; then all solutions of (1.1) are also bounded.

By taking $\alpha(t) \equiv 0$, Theorem 2 and its corollary reduce to Theorem (II); similarly by taking $g(u') \equiv 1$, $f(u) = u^{2n-1}$ where n is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e. $|u(t)| \leqq c\,|t|$ for some positive constant c, and for $t \geqq T$; and the existence of limit of $u'(t)$ as $t \to \infty$.

Theorem 3. Suppose that assumptions A_1, A_2 and A_3 hold and that

(i) $|f(u)| \leqq M\,|u|^{\alpha}$, where M, $\alpha > 0$,

(ii) $\int^{\infty} |a(s)|\, s^{\alpha}\, ds < \infty$,

(iii) $0 < g(v) \leqq K$ for all v;

then the derivative $u'(t)$ of any solution $u(t)$ of (1.1) has a limit if the initial conditions satisfy: for $\alpha > 1$,

$$\left\{ KM(\alpha - 1) \int_{t_0}^{\infty} s^{\alpha}\, |a(s)|\, ds \right\}^{\frac{1}{1-a}} \geqq \left\{ |u(t_0)| + |u'(t_0)| \right\} \tag{2.5}$$

[1]) $a(t)\, F(u(t)) < \infty$ means that a solution $u(t)$ of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy $a(t) F(u(t)) < \infty$. (Note that $a(t) \to O$, as $t \to \infty$ and assumption A_4.)

Proof. Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^{t} (t-s)\, a(s) f(u(s))\, g(u'(s))\, ds .$$

From the hypothesis of the theorem, we have for $t \geqq t_0 \geqq 1$ the following estimate:

$$\frac{|u(t)|}{t} \leqq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^{t} s^{\alpha} KM\, |a(s)| \left(\frac{|u(s)|}{s}\right)^{\alpha} ds \qquad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for $t \geqq t_0 \geqq 1$,

$$\frac{|u(t)|}{t} \leqq \left\{(|u(t_0)| + |u'(t_0)|)^{1-\alpha} + KM(1-\alpha)\int_{t_0}^{t} s^{\alpha}\, |a(s)|\, ds\right\}^{\frac{1}{1-a}} \qquad (2.7)$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^{t} a(s) f(u(s))\, g(u'(s))\, ds$$

and that

$$\left|\int_{t}^{t_0} a(s) f(u(s))\, g(u'(s))\, ds\right| \leqq MK \int_{t}^{t_0} |a(s)\, u^{\alpha}(s)|\, ds$$

$$\leqq MKC^{\alpha} \int_{t_0}^{t} |a(s)|\, s^{\alpha}\, ds$$

where C denotes the bound given in (2.7); we conclude that the limit $\lim\limits_{t \to \infty} u'(t) = \mathrm{L}$ exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

Theorem 4. Suppose that assumptions A_1, A_2 and A_3 hold and in addition that

(a) $|f(u)| \leqslant M\, h(|u|)$,

where $M > O$ aud $h(r)$ is a non-decreasing continous function such that $h(\lambda r) \leqq \lambda^{\alpha} h(r)$, where λ is positive and α is a positive constant; and

$$H(x) = \int_{\infty}^{x} \frac{dr}{h(r)} \to \infty \text{ as } x \to \infty,$$

(b) $\int | a(s) | s^{\alpha} ds < \infty$,

(c) $0 < g(v) \leqq K$ for all v;

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leqq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^{t} s^{\alpha} KM \, |a(s)| \, h\left(\frac{|u(s)|}{s}\right) ds ,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leqq H^{-1}\left(H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^{t} |a(s)| \, s^{\alpha} ds\right)$$

which is bounded for t on account of assumption *(a)*. The remaining proof follows verbatim that of Theorem 3.

3. Boundedness Theorems II

Theorem 5. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that

(i) $a(t) > 0, \quad a'(t) \geqq 0, \quad$ for $\quad t \geqq T$,

(ii) $\dfrac{d}{dt}\left(\dfrac{b}{a}\right) \leqq \beta(t)\left(1 + \dfrac{b}{a}\right)$, with $\int^{\infty} \beta(s) \, ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \geqq \varepsilon > 0 ;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int^{t} \sqrt{a(s)} \, ds$ which tends to infinity as $t \to \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2}\left(\frac{a}{a^{3/2}}\right)\frac{du}{dx} + \left(1 + \frac{b}{a}\right) f(u) \, g(u') = 0 \qquad (3.1)$$

where " dot " denotes differentiation with respect to t. Now write equation in its system form, letting $y_1 = u$:

$$\begin{cases} \dfrac{d\,y_1}{dx} = y_2 \\[2ex] \dfrac{d\,y_2}{dx} = -\dfrac{1}{2}\left(\dfrac{a}{a^{3/2}}\right) y_2 - \left(1 + \dfrac{b}{a}\right) f(y_1)\, g(\sqrt{a}\, y_2) . \end{cases} \tag{3.2}$$

Define for (3.2) the following function:

$$V(x, y_1, y_2) = \left(1 + \frac{b}{a}\right) \int^{y_1} f(s)\, ds + \int^{y_2} \frac{s\, ds}{g(\sqrt{a}\, s)} ,$$

and observe:

$$\begin{aligned} \frac{dV}{dx} &\leqq \frac{\beta(t)}{\sqrt{a(t)}} \left(1 + \frac{b}{a}\right) \int^{y_1} f(s)\, ds - \frac{1}{2} \frac{a}{a^{3/2}} y_2{}^2 \\ &\leqq \frac{\beta(t)}{\sqrt{a(t)}} V . \end{aligned}$$

Hence we have

$$V(x, y_1, y_2) \leqq V\big(x(T),\ y_1(x(T)),\ y_2(x(T)\big) \exp \int_T^t \beta(s)\, ds$$

which is finite. From (ii) we note that $V \to \infty$ as $y_1 \to \infty$ and $V > 0$ if $y_1^2 + y_2^2 \neq 0$. Thus, every solution of (1.1) is bounded.

Corollary. Suppose in addition to the hypothesis of Theorem 5 that assumption A_5 also holds and that $\lim\limits_{t \to \infty} a(t) = a_1 < \infty$, then every solution of (1.1) and its derivative are bounded.

From the above result we may conclude for example that all solutions of the following equation:

$$u''(t) + (c_1 t^\alpha + c_2 t^\beta)\, u^\lambda(t) \big(1 + \exp u'(t) \sin u'(t)\big) = 0$$

are bounded for all c_1, $c_2 > 0$, $\alpha > \beta \geqq 0$, and $\lambda > 0$.

We now consider the following inhomogeneous equation:

$$u''(t) + a(t) f(u)\, g(u') = h(t, u, u') \tag{3.3}$$

and assume that $| u' h(t, u, u') | \leqq \gamma(t)\, g(u')$ where $\int^\infty \gamma(s)\, ds < \infty$.

Theorem 6. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geqq 0$ for $t \geqq T$; then all solutions of (3.3) are bounded.

Proof. Integrate (3.3) in the following manner:

$$G(u'(t)) - G(u'(t_0)) + a(t) F(u(t)) - a(t_0) F(u(t_0))$$

$$= \int_{t_0}^{t} a'(s) F(u(s)) ds + \int_{t_0}^{t} \frac{h(t, u, u') u'(s) ds}{g(u')} \tag{3.4}$$

where $G(v) = \int_0^{\gamma} \frac{s\,ds}{g(s)}$ and $F(u) = \int_0^u f(s)\,ds$. Taking absolute values and noting that $G(v) \geqq 0$ and $F(u) \geqq 0$, we obtain

$$a(t) F(u(t)) \leqq c_0 + c_1 + \int_{t_0}^{t} a'(s) F(u(s)) ds \tag{3.5}$$

where $c_0 = G(u'(t_0)) + a(t_0) F(u(t_0\ \mathrm{O})$ and $c_1 = \int_{t_0}^{\infty} \gamma(s)\,ds$ are non-negative constants. From (3.5) and A_4 it is now clear that every solution of (3.3) are bounded (cf. [1]).

Corollary. In addition to the hypothesis of Theorem 6, suppose that assumption A_5 also holds and that $\lim_{t \to \infty} a(t) = k > 0$; then all solutions of (3.3) and their derivatives are bounded.

We note that by setting $h(t, u, u') \equiv 0$, the above result again reduces to Theorem 1 and its corollary. Other comparison theorems may be formulated in a similar way as Theorem 6 by extending the corresponding result for the homogeneous equation. Since the procedure is clear, the statements and proofs of these results will be omitted.

RÉFÉRENCES

[1] J. S. W. WONG, A note on boundedness theorems of certain second order differential equations. *SIAM Review*, 6 (1964), 175-176.

[2] —— and T. A. BURTON, Some properties of solutions of $u''(t) + a(t) f(u) g(u') = 0$ (II). *Monatshefte für Mathematik*, 69 (1965), 368-374.

[3] J. S. W. WONG, Some properties of solutions of $u''(t) + a(t) f(u) g(u) = 0$ (III). *SIAM Journal*, 14 (1966), 209-214.

[4] ZHANG BING-GEN (CHANG PING-KEN), Boundedness of solutions of ordinary differential equations of the second order. *Acta Math. Sinica*, 14 (1964), 128-137; English translation, *Chinese Mathematics*, 4 (1964), 139-148.

[5] C. T. Taam, Criteria of boundedness of the solutions of non-linear differential equations. *Proc. Amer. Math. Soc.*, 6 (1955), 377-385.
[6] I. Bihari, Researches of the boundedness and stability of the solutions of non-linear differential equations, *Acta. Math. Acad. Sci. Hungar.*, 8 (1957), 261-278.
[7] W. R. Utz, Properties of solutions of $u'' + g(t)\, u^{2n-1} = 0$. *Monatshefte für Mathematik*, 66 (1962), 55-60.
[8] K. M. Das, Properties of solutions of certain non-linear differential equations. *J. of Math. Ana. and Appl.*, 8 (1964), 445-452.
[9] C. M. Petty and G. Leitmann, A boundedness theorem for a non-linear, non-autonomous system. *Monatshefte für Mathematik*, 68 (1964), 46-51.
[10] R. Bellman, *Stability theorem of differential equations.* McGraw-Hill, New York (1953).
[11] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Hung.*, 7 (1956), 81-94.
[12] E. F. Beckenbach and R. Bellman, *Inequalities.* Springer-Verlag, Berlin (1961).
[13] Ju. A. Klokov, Some theorems on boundedness of solutions of ordinary differential equations. *Uspehi Mat. Nauk*, 13 (1958), No. 2 (80), 189-194; English translation. *Amer. Math. Soc. Transl.*, 18 (1961), 289-294.
[14] P. Waltman, Some properties of solutions of $u''(t) + a(t) f(u) = 0$. *Monatshefte für Mathematik*, 67 (1963), 50-54.
[15] Li Yue-sheng (Li Yueh-sheng), The bound, stability, and error estimates for the solution of non-linear differential equations. *Acta Math. Sinica*, 12 (1962), 32-39; English translation, *Chinese Mathematics*, 3 (1963), 34-41.
[16] D. Willett and J. S. W. Wong, On the discrete analogues of some generalizations of Gronwall's inequality. *Monatshefte für Mathematik*, 69 (1965), 362-367.
[17] J. S. W. Wong, On two theorems of Waltman. *SIAM Journal*, 14 (1966), 724-728.
[18] P. Waltman, On the asymptotic behaviour of solutions of a non-linear equation. *Proc. Amer. Math. Soc.*, 15 (1964), 918-923.

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