

## 3. BOUNDEDNESS THEOREMS II

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$$(b) \int |a(s)| s^\alpha ds < \infty,$$

$$(c) 0 < g(v) \leq K \text{ for all } v;$$

then the derivative of any solution of (1.1) has a limit.

*Proof.* Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leq H^{-1} (H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^t |a(s)| s^\alpha ds)$$

which is bounded for  $t$  on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

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*Theorem 5.* Suppose that assumptions  $A_1, A_2, A_3$  and  $A_4$  hold and in addition that

$$(i) a(t) > 0, \quad a'(t) \geq 0, \quad \text{for } t \geq T,$$

$$(ii) \frac{d}{dt} \left( \frac{b}{a} \right) \leq \beta(t) \left( 1 + \frac{b}{a} \right), \quad \text{with } \int_0^\infty \beta(s) ds < \infty$$

and

$$\left( 1 + \frac{b}{a} \right) \geq \varepsilon > 0;$$

then every solution of (1.1) with  $(a(t) + b(t))$  replacing  $a(t)$  is bounded.

*Proof.* Make the following substitution for the independent variable,  $x = \int_0^t \sqrt{a(s)} ds$  which tends to infinity as  $t \rightarrow \infty$ , and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left( \frac{a}{a^{3/2}} \right) \frac{du}{dx} + \left( 1 + \frac{b}{a} \right) f(u) g(u') = 0 \quad (3.1)$$

where “dot” denotes differentiation with respect to  $t$ . Now write equation in its system form, letting  $y_1 = u$ :

$$\begin{cases} \frac{d y_1}{d x} = y_2 \\ \frac{d y_2}{d x} = -\frac{1}{2} \left( \frac{a}{a^{3/2}} \right) y_2 - \left( 1 + \frac{b}{a} \right) f(y_1) g(\sqrt{a} y_2). \end{cases} \quad (3.2)$$

Define for (3.2) the following function:

$$V(x, y_1, y_2) = \left( 1 + \frac{b}{a} \right) \int^{y_1} f(s) ds + \int^{y_2} \frac{s ds}{g(\sqrt{a} s)},$$

and observe:

$$\begin{aligned} \frac{dV}{dx} &\leq \frac{\beta(t)}{\sqrt{a(t)}} \left( 1 + \frac{b}{a} \right) \int^{y_1} f(s) ds - \frac{1}{2} \frac{a}{a^{3/2}} y_2^2 \\ &\leq \frac{\beta(t)}{\sqrt{a(t)}} V. \end{aligned}$$

Hence we have

$$V(x, y_1, y_2) \leq V(x(T), y_1(x(T)), y_2(x(T))) \exp \int_T^t \beta(s) ds$$

which is finite. From (ii) we note that  $V \rightarrow \infty$  as  $y_1 \rightarrow \infty$  and  $V > 0$  if  $y_1^2 + y_2^2 \neq 0$ . Thus, every solution of (1.1) is bounded.

*Corollary.* Suppose in addition to the hypothesis of Theorem 5 that assumption  $A_5$  also holds and that  $\lim_{t \rightarrow \infty} a(t) = a_1 < \infty$ , then every solution of (1.1) and its derivative are bounded.

From the above result we may conclude for example that all solutions of the following equation:

$$u''(t) + (c_1 t^\alpha + c_2 t^\beta) u^\lambda(t) (1 + \exp u'(t) \sin u'(t)) = 0$$

are bounded for all  $c_1, c_2 > 0$ ,  $\alpha > \beta \geq 0$ , and  $\lambda > 0$ .

We now consider the following inhomogeneous equation:

$$u''(t) + a(t) f(u) g(u') = h(t, u, u') \quad (3.3)$$

and assume that  $|u' h(t, u, u')| \leq \gamma(t) g(u')$  where  $\int_0^\infty \gamma(s) ds < \infty$ .

*Theorem 6.* Suppose that assumptions  $A_1, A_2, A_3$  and  $A_4$  hold and in addition that  $a(t) > 0$  and  $a'(t) \geq 0$  for  $t \geq T$ ; then all solutions of (3.3) are bounded.

*Proof.* Integrate (3.3) in the following manner:

$$G(u'(t)) - G(u'(t_0)) + a(t)F(u(t)) - a(t_0)F(u(t_0)) \\ = \int_{t_0}^t a'(s)F(u(s))ds + \int_{t_0}^t \frac{h(t, u, u')u'(s)ds}{g(u')} \quad (3.4)$$

where  $G(v) = \int_0^v \frac{s ds}{g(s)}$  and  $F(u) = \int_0^u f(s) ds$ . Taking absolute values and noting that  $G(v) \geq 0$  and  $F(u) \geq 0$ , we obtain

$$a(t)F(u(t)) \leq c_0 + c_1 + \int_{t_0}^t a'(s)F(u(s))ds \quad (3.5)$$

where  $c_0 = G(u'(t_0)) + a(t_0)F(u(t_0))$  and  $c_1 = \int_{t_0}^{\infty} \gamma(s) ds$  are non-negative constants. From (3.5) and  $A_4$  it is now clear that every solution of (3.3) are bounded (cf. [1]).

*Corollary.* In addition to the hypothesis of Theorem 6, suppose that assumption  $A_5$  also holds and that  $\lim_{t \rightarrow \infty} a(t) = k > 0$ ; then all solutions of (3.3) and their derivatives are bounded.

We note that by setting  $h(t, u, u') \equiv 0$ , the above result again reduces to Theorem 1 and its corollary. Other comparison theorems may be formulated in a similar way as Theorem 6 by extending the corresponding result for the homogeneous equation. Since the procedure is clear, the statements and proofs of these results will be omitted.

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