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A NATURAL SETTING FOR THE EXTENSIONS OF A GROUP WITH TRIVIAL CENTRE BY AN ARBITRARY GROUP

by John S. ROSE

Let A and H be groups. The intention of the present note is to point out that if A has trivial centre, then any extension of A by H is equivalent (in the sense of extension theory) to one determined in a natural way by a suitable subgroup of $\text{Aut } A \times H$. This fact is already implicit in an early paper of R. BAER [1]¹, but the aim here is to provide a more explicit formulation; and to deduce that the non-equivalent extensions of a group A with trivial centre by an arbitrary group H stand in one-to-one correspondence with the distinct homomorphisms of H into the group $\text{Aut } A/\text{Inn } A$ of automorphism classes of A . This latter result is obtained in the treatment of KUROSH [5, p. 148] as a corollary of some cohomological theorems of S. EILENBERG and S. MACLANE [3]. The proof offered here is an entirely elementary application of the fact that it is possible to work within $\text{Aut } A \times H$.

The notation and terminology used are largely standard. For an arbitrary group A , $\text{Aut } A$ denotes the group of all automorphisms of A and $\text{Inn } A$ the group of all inner automorphisms of A . We shall denote by $\mathfrak{U}(A)$ the group $\text{Aut } A/\text{Inn } A$ of automorphism classes of A . If B is a subgroup of A , $C_A(B)$ denotes the centralizer of B in A . Then $C_A(A) = Z(A)$, the centre of A . For an arbitrary element a of A , the inner automorphism of A induced by a is denoted by τ_a ; this notation is relative to the group A , which is here presumed to be fixed. The groups $A/Z(A)$ and $\text{Inn } A$ may be identified in the natural way by identifying, for each a in A , the elements $aZ(A)$ and τ_a ; this identification will be made. An automorphism α of A is called a central automorphism if, for every a in A , $(a\alpha)a^{-1} \in Z(A)$. It is easy to show that the set of all central automorphisms of A forms a subgroup of $\text{Aut } A$ which is in fact precisely $C_{\text{Aut } A}(\text{Inn } A)$: see ZASSENHAUS [6, p. 52].

¹) See also H. FITTING [4, § 21].

Let A and H be arbitrary groups. An *extension* of A by H is a pair (G, φ) consisting of a group G , containing A as a normal subgroup, and a homomorphism φ of G onto H such that $\text{Ker } \varphi = A$. (Reference to the particular homomorphism φ involved in an extension is often omitted, for instance in KUROSH [5, Chapter XII], but φ is tacitly assumed to be specified in the development of the theory.) Two extensions (G, φ) and (G^*, φ^*) of A by H are said to be *equivalent* if there is an isomorphism Θ of G onto G^* mapping A identically onto itself and such that $\Theta\varphi^* = \varphi$.

Suppose that (G, φ) is an extension of A by H , and that B is a characteristic subgroup of A . Then B is a normal subgroup of G , and (G, φ) induces naturally an extension $(G/B, \bar{\varphi})$ of A/B by H : $\bar{\varphi}$ is defined by

$$(gB)\bar{\varphi} = g\varphi, \quad \text{for any } g \text{ in } G;$$

this is well defined since $B \leq A = \text{Ker } \varphi$. We shall be concerned with the special case in which $B = Z(A)$.

Let $\bar{A} = A/Z(A)$. It is possible, for arbitrary groups A and H , to construct extensions of \bar{A} by H rather transparently by means of suitable subgroups of $\text{Aut } A \times H$ (the external direct product). $\text{Aut } A$ is identified with a subgroup of this direct product in the obvious way by identification of α with $(\alpha, 1)$, for each α in $\text{Aut } A$. Then \bar{A} , which is identified with $\text{Inn } A$, is also identified with a subgroup of $\text{Aut } A \times H$. Let π denote the projection homomorphism of $\text{Aut } A \times H$ onto H :

$$(\alpha, h)\pi = h, \quad \text{for any } \alpha \text{ in } \text{Aut } A \text{ and } h \text{ in } H.$$

Then any subgroup Q of $\text{Aut } A \times H$ such that $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$ determines an extension (Q, π_0) of \bar{A} by H , where π_0 is simply the restriction of π to Q : for π_0 is a homomorphism of Q onto H , since $Q\pi = H$, and $\text{Ker } \pi_0 = Q \cap \text{Ker } \pi = Q \cap \text{Aut } A = \bar{A}$. For convenience, we introduce a term for such an extension: we shall call it a *sited extension* of \bar{A} by H .

We shall prove the

THEOREM. *Let A and H be arbitrary groups. Suppose that (G, φ) is an extension of A by H , and let $(\bar{G}, \bar{\varphi})$ be the induced extension of \bar{A} by H , where $\bar{G} = G/Z(A)$, $\bar{A} = A/Z(A)$. Then $(\bar{G}, \bar{\varphi})$ is equivalent to a sited extension of \bar{A} by H . Moreover, if the only central automorphisms of A are*

inner automorphism, then sited extensions of \bar{A} by H corresponding to distinct subgroups of $\text{Aut } A \times H$ are non-equivalent.

Proof. For any element g of G , let σ_g denote the restriction to A of the inner automorphism of G induced by g . (Thus $\sigma_a = \tau_a$, for each a in A .) We define a map $\psi: G \rightarrow \text{Aut } A \times H$ by

$$g\psi = (\sigma_g, g\varphi), \quad \text{for every } g \text{ in } G.$$

Clearly ψ is a homomorphism, and

$$\begin{aligned} \text{Ker } \psi &= \{ g \in G \mid g^{-1}ag = a \text{ for all } a \text{ in } A \} \cap \text{Ker } \varphi \\ &= C_G(A) \cap A \\ &= Z(A). \end{aligned}$$

Then ψ induces naturally an isomorphism $\bar{\psi}$ of \bar{G} onto a subgroup Q of $\text{Aut } A \times H$; and

$$\begin{aligned} Q \cap \text{Aut } A &= \{ (\sigma_g, g\varphi) \mid g \in G, g\varphi = 1 \} \\ &= \{ (\sigma_g, 1) \mid g \in \text{Ker } \varphi \} \\ &= \bar{A}, \\ Q\pi &= \{ g\varphi \mid g \in G \} \\ &= \text{Im } \varphi \\ &= H. \end{aligned}$$

Hence Q determines a sited extension (Q, π_0) of \bar{A} by H , where π_0 is the restriction of π to Q .

We show that $(\bar{G}, \bar{\varphi})$ is equivalent to (Q, π_0) . For this purpose we can use $\bar{\psi}$, which is an isomorphism of \bar{G} onto Q . For any element g of G , let $\bar{g} = gZ(A)$. Then

$$\bar{g}(\bar{\psi}\pi_0) = (g\psi)\pi_0 = (\sigma_g, g\varphi)\pi = g\varphi = \bar{g}\bar{\varphi},$$

so that

$$\bar{\psi}\pi_0 = \bar{\varphi}.$$

Also, for any a in A ,

$$\bar{a}\bar{\psi} = a\psi = (\sigma_a, a\varphi) = (\tau_a, 1) = \bar{a},$$

by identification, so that $\bar{\psi}$ maps \bar{A} identically onto itself. This establishes the equivalence of $(\bar{G}, \bar{\varphi})$ and (Q, π_0) .

Now assume that the only central automorphisms of A are inner, that is that $C_{\text{Aut } A}(\bar{A}) \leq \bar{A}$. Suppose that Q, Q^* are subgroups of $\text{Aut } A \times H$ such that $Q \cap \text{Aut } A = \bar{A} = Q^* \cap \text{Aut } A$ and $Q\pi = H = Q^*\pi$, so that Q, Q^* determine sited extensions $(Q, \pi_0), (Q^*, \pi_0^*)$ of \bar{A} by H . Suppose that these extensions are equivalent. Then there is an isomorphism Θ of Q onto Q^* mapping \bar{A} identically onto itself and such that $\Theta\pi_0^* = \pi_0$.

For each h in H , we choose α_h in $\text{Aut } A$ such that $(\alpha_h, h) \in Q$: this is possible since $Q\pi = H$. (In general h does not determine a unique such element α_h , but we make a choice of one element for each h .) Let

$$(\alpha_h, h) \Theta = (\alpha_h^*, h^*), \quad \text{with } \alpha_h^* \text{ in } \text{Aut } A \text{ and } h^* \text{ in } H.$$

Since \bar{A} is a normal subgroup of $\text{Aut } A$,

$$\alpha_h^{-1} \bar{a} \alpha_h \in \bar{A} \quad \text{for any } \bar{a} \text{ in } \bar{A},$$

and therefore

$$(\alpha_h^{-1} \bar{a} \alpha_h) \Theta = \alpha_h^{-1} \bar{a} \alpha_h. \quad (1)$$

Now (by identification)

$$\alpha_h^{-1} \bar{a} \alpha_h = (\alpha_h, h)^{-1} \bar{a} (\alpha_h, h). \quad (2)$$

Since (α_h, h) and \bar{a} both belong to Q , (1) and (2) give

$$\begin{aligned} \alpha_h^{-1} \bar{a} \alpha_h &= ((\alpha_h, h) \Theta)^{-1} (\bar{a} \Theta) ((\alpha_h, h) \Theta) \\ &= (\alpha_h^*, h^*)^{-1} \bar{a} (\alpha_h^*, h^*), \end{aligned}$$

that is

$$\alpha_h^{-1} \bar{a} \alpha_h = (\alpha_h^*)^{-1} \bar{a} \alpha_h^*. \quad (3)$$

Hence $\alpha_h^* \alpha_h^{-1} \in C_{\text{Aut } A}(\bar{A}) \leq \bar{A}$, by hypothesis. Thus for each h in H , there is an element η_h in \bar{A} such that

$$\alpha_h^* = \eta_h \alpha_h. \quad (4)$$

Also

$$h^* = (\alpha_h, h) \Theta \pi_0^* = (\alpha_h, h) \pi_0 = h,$$

so that

$$(\alpha_h, h) \Theta = (\alpha_h^*, h),$$

that is

$$(\alpha_h, h) \Theta = \eta_h (\alpha_h, h). \quad (5)$$

Now we consider an arbitrary element of Q , say (α, h) with α in $\text{Aut } A$ and h in H . Since also $(\alpha_h, h) \in Q$ and $Q \cap \text{Aut } A = \bar{A}$, there is an element \bar{a} in \bar{A} such that

$$(\alpha, h) = \bar{a}(\alpha_h, h).$$

Then

$$\begin{aligned} (\alpha, h) \Theta &= (\bar{a} \Theta)((\alpha_h, h) \Theta) \\ &= \bar{a} \eta_h(\alpha_h, h), \quad \text{by (5).} \end{aligned}$$

Since $\bar{A} \leq Q$, this shows that $(\alpha, h) \Theta \in Q$. Hence $Q^* = Q\Theta \leq Q$. Similarly $Q \leq Q^*$. Therefore $Q = Q^*$. This completes the proof.

We observe now that the distinct subgroups of $\text{Aut } A \times H$ determining sited extensions of \bar{A} by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}(A)$. To see this, suppose first that Q is a subgroup of $\text{Aut } A \times H$ determining a sited extension of \bar{A} by H , that is such that $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$. Then Q determines a homomorphism $\lambda_Q: H \rightarrow \mathfrak{A}(A)$ as follows:

$$\text{for any } h \text{ in } H, h\lambda_Q = \alpha\bar{A} \text{ if and only if } (\alpha, h) \in Q,$$

where $\alpha \in \text{Aut } A$.

Since $Q \cap \text{Aut } A = \bar{A}$, λ_Q is well defined by this rule, and is defined on the whole of H since $Q\pi = H$. Conversely, suppose that λ is a homomorphism of H into $\mathfrak{A}(A)$. Then λ determines a subgroup Q of $\text{Aut } A \times H$, defined as

$$Q = \{ (\alpha, h) \mid \alpha \in \text{Aut } A, h \in H \text{ and } h\lambda = \alpha\bar{A} \},$$

and it is clear that then $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$, so that Q determines a sited extension of \bar{A} by H . Furthermore, $\lambda_Q = \lambda$. Finally, distinct homomorphisms of H into $\mathfrak{A}(A)$ evidently determine distinct subgroups of $\text{Aut } A \times H$, and so the correspondence between homomorphisms and subgroups is one-to-one.

If A is a group with trivial centre, then A is naturally identified with \bar{A} and the Theorem shows that any extension of A by H is equivalent to a sited extension of A by H . Moreover, the only central automorphism of A is the identity automorphism, so that we obtain

COROLLARY 1. *Let A be a group with trivial centre and H an arbitrary group. Then every extension of A by H is equivalent to a sited extension*

of A by H . The non-equivalent extensions of A by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{U}(A)$.

If the only homomorphism of H into $\mathfrak{U}(A)$ is the trivial homomorphism, then the only sited extension of \bar{A} by H is $(\bar{A} \times H, \pi)$, where π denotes the projection map of $\bar{A} \times H$ onto H . Thus in particular we have

COROLLARY 2. Let A be a group with trivial centre and H a group. Then (up to equivalence) the only extension of A by H is $(A \times H, \pi)$, where π denotes the projection map of $A \times H$ onto H , in any of the following cases:

- (i) $\mathfrak{U}(A)$ is trivial.
- (ii) $\mathfrak{U}(A)$ is soluble and H is perfect.
- (iii) $\mathfrak{U}(A)$ is a ϖ -group and H is a ϖ' group, where ϖ is a set of prime numbers and ϖ' the set of all prime numbers not belonging to ϖ .
- (iv) H is simple and cannot be embedded in $\mathfrak{U}(A)$.

Here (i) is the well known case of a complete group A .

According to a celebrated conjecture of O. SCHREIER, $\mathfrak{U}(E)$ ought to be soluble for any finite non-abelian simple group E . SCHREIER's Conjecture is valid for every known finite non-abelian simple group. Thus (ii) applies if A is any known finite non-abelian simple group.

Another result can be derived from (ii) and a Lemma due to H. FITTING [4, Satz 12], which may be expressed as follows.

LEMMA. Let E be a finite non-abelian simple group. Then, if n is a positive integer and D is the direct product of n copies of E , $\text{Aut } D$ is isomorphic to the wreath product of $\text{Aut } E$ by the symmetric group of degree n , formed according to the natural representation.

A group is called *completely reducible* if it can be decomposed as a direct product of a finite number of simple groups (KUROSH [5, p. 203]). An easy inductive proof, using (ii) and the Lemma, establishes

COROLLARY 3. Let G be a non-trivial finite group. Associated with G there is a set of non-isomorphic simple groups E_1, \dots, E_k and a set of positive integers n_1, \dots, n_k such that every composition series of G has precisely n_i composition factors isomorphic to E_i ($i = 1, \dots, k$) and no others. If every E_i is non-abelian and satisfies SCHREIER's Conjecture, and if every $n_i < 5$, then G is completely reducible.

This is a particular case of a recent result of R. BERCOV [2].

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