

# CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

Autor(en): **Mott, Thomas E.**

Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

Persistenter Link: <https://doi.org/10.5169/seals-42354>

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

# CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

by Thomas E. MOTT

In a recent paper [1], R. L. Kruse and J. J. Deeley have proved an interesting theorem concerning the continuity of a real valued function of several real variables, when that function is continuous in each variable separately and satisfies a certain monotonicity condition. The proof given by Kruse and Deeley involves induction on the variables, however a somewhat shorter and simpler proof is given below. In addition, two interesting corollaries are stated.

**THEOREM 1.** — *Let  $f(x_1, \dots, x_n)$  be a real valued function defined on an open set  $G \subseteq R^n$ , and suppose that :*

- (i) *Whenever  $n - 1$  of the variables are fixed,  $f$  is a continuous function of the remaining variable.*
- (ii) *For each permissible <sup>1)</sup> value of  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  in  $R^{n-1}$  the function  $f(x_1, \dots, x_n)$  is a monotone function of  $x_i$ , the direction of monotonicity being dependent upon the choice of the point  $(x_1, \dots, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  in  $R^{n-1}$ ; all for  $i = 1, \dots, n$ .  
Then  $f(x_1, \dots, x_n)$  is continuous in  $G$ .*

*Proof:* Let  $(x_{1,0}, \dots, x_{n,0})$  be any point in  $G$ , then  $G$  being an open set we may choose  $\delta > 0$  such that the rectangle  $S = [x_{1,0} - \delta, x_{1,0} + \delta] \times \dots \times [x_{n,0} - \delta, x_{n,0} + \delta]$  is contained in  $G$ . In view of (i), given  $\epsilon > 0$  we may choose  $\delta_1$  in  $(0, \delta)$  such that

$$|f(x_1, x_{2,0}, \dots, x_{n,0}) - f(x_{1,0}, x_{2,0}, \dots, x_{n,0})| < \frac{\epsilon}{n}$$

whenever  $|x_1 - x_{1,0}| \leq \delta_1$ ,  $\delta_2$  in  $(0, \delta_2)$  such that

$$|f(x_{1,0} \pm \delta_1, x_2, x_3, \dots, x_{n,0}) - f(x_{1,0} \pm \delta_1, x_{2,0}, x_3, \dots, x_{n,0})| < \frac{\epsilon}{n}$$

---

<sup>1)</sup> Permissible values of  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$  in  $R^{n-1}$  being those for which  $(x_1, \dots, x_n) \in G$ .

whenever  $|x_2 - x_{2,0}| \leq \delta_2$ , and continuing in this manner we finally choose  $\delta_n$  in  $(0, \delta)$  such that

$$|f(x_{1,0} \pm \delta_1, x_{2,0} \pm \delta_2, \dots, x_{n-1,0} \pm \delta_{n-1}, x_n) - f(x_{1,0} \pm \delta_1, x_{2,0} \pm \delta_2, \dots, x_{n-1,0} \pm \delta_{n-1}, x_{n,0})| < \frac{\varepsilon}{n}$$

whenever  $|x_n - x_{n,0}| \leq \delta_n$ .

Let  $\bar{S} = [x_{1,0} - \delta_1, x_{1,0} + \delta_1] \times \dots \times [x_{n,0} - \delta_n, x_{n,0} + \delta_n]$ , then romf (ii) it follows that the function  $f$  assumes its maximum and minimum values at vertices of  $\bar{S}$ , let  $(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*)$  and  $(x_{1,0} + \delta_1^{**}, \dots, x_{n,0} + \delta_n^{**})$  be these maximum and minimum points respectively, then  $\delta_i^* = \pm \delta_i$  and  $\delta_i^{**} = \delta_i$  for  $i = 1, \dots, n$  and certain choices of the  $\pm$  signs.

Now

$$\begin{aligned} & |f(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*) - f(x_{1,0}, \dots, x_{n,0})| \leq \\ & |f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0} + \delta_n^*) - \\ & - f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0})| + \\ & + |f(x_{1,0} + \delta_1^*, \dots, x_{n-1,0} + \delta_{n-1}^*, x_{n,0}) - \\ & - f(x_{1,0} + \delta_1^*, \dots, x_{n-2,0} + \delta_{n-2}^*, x_{n-1,0}, x_{n,0})| + \\ & + \dots + |f(x_{1,0} + \delta_1^*, x_{2,0}, \dots, x_{n,0}) - f(x_{1,0}, \dots, x_{n,0})| < \varepsilon \end{aligned}$$

and similarly  $|f(x_{1,0} + \delta_1^{**}, \dots, x_{n,0} + \delta_n^{**}) - f(x_{1,0}, \dots, x_{n,0})| < \varepsilon$ . Therefore,  $|f(x_{1,0} + \delta_1^*, \dots, x_{n,0} + \delta_n^*) - f(x_{1,0} + \delta_1^{**}, \dots, x_{n,0} + \delta_n^{**})| < 2\varepsilon$  and consequently if  $(x'_1, \dots, x'_n)$ ,  $(x''_1, \dots, x''_n)$  are any two points of  $\bar{S}$  then  $|f(x'_1, \dots, x'_n) - f(x''_1, \dots, x''_n)| < 2\varepsilon$ . Since  $\varepsilon$  is arbitrary it now follows from the Cauchy Criterion that the function  $f$  is continuous at the point  $(x_{1,0}, \dots, x_{n,0})$  in  $G$ .

Two rather interesting results which follow directly from this theorem are:

*Corollary 1:* Let  $f(x_1, \dots, x_n)$  be a real valued function defined on an open set  $G \subseteq R^n$ . Let  $T$  be an invertible mapping from  $G$  into  $R^n$  defined by the equations  $u_i = p_i(x_1, \dots, x_n)$  ( $i=1, \dots, n$ ) in such a manner that the inverse mapping  $T^{-1}$  is defined by the equations  $x_i = q_i(u_1, \dots, u_n)$  ( $i=1, \dots, n$ ) where the functions  $p_i(x_1, \dots, x_n)$  ( $i=1, \dots, n$ ) are continuous in  $G$  and the functions  $q_i(u_1, \dots, u_n)$  ( $i=1, \dots, n$ ) are continuous in  $T(G)$ . Suppose that:

- (i) The function  $f$  is continuous along that portion of the curves  $\{ x_1 = q_1(u_1+t, u_2, \dots, u_n), \dots, x_n = q_n(u_1+t, u_2, \dots, u_n) \}, \dots, \{ x_1 = q_1(u_1, \dots, u_{n-1}, u_n+t), \dots, x_n = q_n(u_1, \dots, u_{n-1}, u_n+t) \}$  which lie in  $G$ , for every  $(u_1, \dots, u_n)$  in  $T(G)$ .
- (ii) For each permissible <sup>1)</sup> value of  $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$  in  $R^{n-1}$  the function  $f(q_1(u_1, \dots, u_n), \dots, q_n(u_1, \dots, u_n))$  is a monotonic function of  $u_i$ , the direction of monotonicity being dependent upon the choice of the point  $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$  in  $R^{n-1}$ ; all for  $i = 1, \dots, n$ . Then  $f(x_1, \dots, x_n)$  is continuous in  $G$ .

*Corollary 2:* Let  $f(x_1, \dots, x_n)$  be a real valued function defined on an open set  $G \subseteq R^n$  and let  $v_i = (\lambda_{i,1}, \dots, \lambda_{i,n})$  ( $i=1, \dots, n$ ) be linearly independent vectors in  $R^n$ . If the function  $f$  is continuous along that portion of every line passing through  $G$  and parallel to  $v_i$  ( $i=1, \dots, n$ ), and  $f$  is monotonic along each of these lines (the direction of monotonicity depending upon the choice of line), then  $f(x_1, \dots, x_n)$  is continuous in  $G$ .

#### REFERENCES

- [1] KRUSE, R. L. and J. J. DEELY, "Joint Continuity of Monotonic Functions, *Amer Math. Monthly*, 7» (1969), pg. (74-76).

(Reçu le 1<sup>er</sup> juin 1969)

State University College  
Buffalo, N.Y. 14222

---

<sup>1)</sup> Permissible values of  $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$  in  $R^{n-1}$  being those for which  $(u_1, \dots, u_n) \in T(G)$ .