

# 1.1 Reduced analytic spaces.

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### 1.1 Reduced analytic spaces.

To prepare for the general definition we shall first introduce reduced analytic spaces and their local models. Let  $U$  be an open set in  $\mathbf{C}^n$  and  $V$  an analytic subset of  $U$ . The sheaf  $\mathcal{I}$  on  $U$  of all germs of holomorphic functions vanishing on  $V$  is coherent by the Oka-Cartan theorem (for a proof, see e.g. Narasimhan [9, Theorem 5, p. 77]). The support of  $\mathcal{O}_U/\mathcal{I}$  is  $V$ , and we shall denote by  $\mathcal{O}_V$  the restriction of  $\mathcal{O}_U/\mathcal{I}$  to  $V$  ( $\mathcal{O}_U$  denotes the sheaf on  $U$  of germs of holomorphic functions). The *local models for reduced analytic spaces* shall be the pairs  $(V, \mathcal{O}_V)$ . Obviously we may consider  $\mathcal{O}_V$  as a subsheaf of  $\mathcal{C}_V$ , the sheaf on  $V$  of germs of continuous functions.

*Definition 1.1.1.* A *reduced analytic space* is a pair  $(X, \mathcal{O}_X)$  where  $X$  is a topological space (not necessarily separated) and  $\mathcal{O}_X$  is a sheaf of sub- $\mathbf{C}$ -algebras of  $\mathcal{C}_X$  which is locally isomorphic to a local model.

To be explicit, the last property means that every point  $x \in X$  has a neighborhood  $U$  such that for some local model  $(V, \mathcal{O}_V)$  there is a homeomorphism  $\varphi : U \rightarrow V$  with the property that for  $y \in U$ ,  $f \in \mathcal{C}_{U,y}$  belongs to  $\mathcal{O}_{U,y}$  if and only if  $f = g \circ \varphi$  for some germ  $g \in \mathcal{O}_{V,\varphi(y)}$ .

As a common abuse of language we shall sometimes write  $X$  instead of  $(X, \mathcal{O}_X)$ .

Reduced analytic spaces need not be separated. Consider for example the disjoint union of two copies of  $\mathbf{C}$ , with all points except the origins identified. This topological space is in a natural way a reduced analytic space, indeed a complex manifold.

Reduced analytic spaces were introduced by Cartan-Serre (under the name of “analytic spaces”).

*Definition 1.1.2.* A *morphism, or holomorphic map* of one reduced analytic space  $(X, \mathcal{O}_X)$  into another,  $(Y, \mathcal{O}_Y)$ , is a continuous map  $\varphi : X \rightarrow Y$  such that  $\varphi^*(\mathcal{O}_{Y,\varphi(x)}) \subset \mathcal{O}_X$  for all  $x \in X$ .

This definition, of course, gives us also the notion of isomorphism of reduced analytic spaces, which we have already used in a special case in Definition 1.1.1.

*Example 1.* If  $X, Y$  are complex manifolds, the morphisms of  $(X, \mathcal{O}_X)$  into  $(Y, \mathcal{O}_Y)$  are the holomorphic maps  $X \rightarrow Y$  in the usual sense.

*Example 2.* The morphisms of  $(X, \mathcal{O}_X)$  into  $\mathbf{C}$ , regarded as a reduced analytic space  $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$ , can be identified with the sections  $\Gamma(X, \mathcal{O}_X)$ .

*Example 3.* The morphisms of  $(X, \mathcal{O}_X)$  into  $\mathbf{C}^n$  can be identified with  $n$ -tuples of sections of  $\mathcal{O}_X$ , or, again, with sections of  $\mathcal{O}_X^n$ .

It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map  $t \rightarrow (t^2, t^3)$  of  $X = \mathbf{C}$  into the space  $Y$  of all pairs  $(x, y)$  satisfying  $x^3 - y^2 = 0$ . This is a bijective and bicontinuous morphism, but its inverse  $\psi$  is no morphism since  $\psi^* f_0 \notin \mathcal{O}_{Y,0}$  if  $f(t) = t$ .

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider “Cartan’s umbrella” which is the subset of  $\mathbf{R}^3$  defined by the equation  $z(x^2 + y^2) - x^3 = 0$ . Its intersection with the plane  $z = 1$  has an isolated double point at  $(0, 0, 1)$  and so it has a stick (the  $z$ -axis) joining the rest of the “umbrella” at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf  $\mathcal{S}$  of germs of real-analytic functions vanishing on the umbrella were generated by sections  $s_1, \dots, s_n \in \Gamma(U, \mathcal{S})$  over some neighborhood  $U$  of the origin. Then, denoting by  $f_1, \dots, f_n$  the corresponding real-analytic functions in  $U$ , we find (using a complexification and the Nullstellensatz for principal ideals) that every  $f_j$  is a multiple of  $z(x^2 + y^2) - x^3$  for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in  $\mathcal{S}$  defined by the coordinate function  $x$  at a point  $(0, 0, z)$ ,  $z \neq 0$ , cannot be a linear combination of  $S_1, \dots, S_n$  which is a contradiction.

### 1.2. Definition of general analytic spaces.

Let  $U$  be an open subset of  $\mathbf{C}^n$  (or  $\mathbf{R}^n$ ) and let  $\mathcal{S}$  be an arbitrary coherent sheaf of ideals in  $\mathcal{O}_U$ , the sheaf on  $U$  of germs of holomorphic (or real-analytic) functions. Then  $V = \text{supp } \mathcal{O}_U/\mathcal{S}$  is an analytic subset of  $U$ . The restriction of  $\mathcal{O}_U/\mathcal{S}$  to  $V$  will be denoted by  $\mathcal{O}_V$ . It is, in general, not a subsheaf of  $\mathcal{C}_V$ . The definition of a general analytic space will be based on *local models*  $(V, \mathcal{O}_V)$  of the type just constructed. Note that a model  $(V, \mathcal{O}_V)$  is of the previously considered reduced type if and only if  $\mathcal{S}$  is the sheaf of *all* germs of holomorphic functions vanishing on  $V$ . In the general case the set  $V$  does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

*Example 1.* Let  $U = \mathbf{C}$ ,  $\mathcal{S}$  the sheaf of ideals generated by  $x^2$ . Here  $V = \{0\}$  and  $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$  ( $\mathbf{C}\{x\}$  denotes the space of converging power series in the variable  $x$ ). Thus  $\mathcal{O}_{V,0}$  is the space of “dual numbers” representable as  $a + b\varepsilon$  where  $a, b \in \mathbf{C}$  and  $\varepsilon^2 = 0$ ,  $\varepsilon$  being the class of  $x$ . Evidently  $\mathcal{O}_{V,0}$  cannot be a subring of the continuous functions on  $\{0\}$ . The