1.4. Relations between reduced and non-reduced spaces.

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$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_u\searrow \quad \swarrow_v & \\ & S & \end{array}$$

is commutative. The product in this category, i.e. the object satisfying the universal property given above for the product $X \times Y$, is then exactly the fiber product $X \times_s Y$. If S is a point, we have the category of analytic spaces.

Example 3. If U and V are open subspaces of an analytic space X, the open subspace $U \cap V$ is isomorphic to $U \times_X V$. We may thus define, in general, the intersection of two analytic subspaces $X' \to X$ and $X'' \to X$ of X to be the fiber product $X' \times_X X''$.

Example 4. If $\varphi : Y \to X$ is a morphism of analytic spaces and $a \in X$ a point, i.e. a map $a : (0, \mathbf{C}) \to X$ we may consider the space $Y(a) = Y \times_X a$. It is natural to call this the inverse image of a under φ and to denote it by $\varphi^{-1}(a)$; its underlying space is exactly $\varphi_0^{-1}(a)$.

If $\varphi_0(b) = a$, then $\mathcal{O}_{Y(a),b}$ is $\mathcal{O}_{Y,b}$ taken modulo the image under $\varphi^1 : \mathcal{O}_{X,a} \to \mathcal{O}_{Y,b}$ of the maximal ideal in $\mathcal{O}_{X,a}$.

Example 5. The pull-back of a linear bundle E over X by a map $Y \to X$ is exactly $Y \times_X E$.

1.4. *Relations between reduced and non-reduced spaces.*

We shall first characterize those analytic spaces which are reduced.

Proposition 1.4.1. A analytic space $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent element for x arbitrary in X.

Proof. The necessity of the condition is obvious for $\mathcal{O}_X$ can be considered as a submodule of $\mathscr{C}_X$ if $(X, \mathcal{O}_X)$ is reduced.

Conversely, if $\mathcal{O}_{X,x}$ has no nilpotent elements, we shall prove that in any local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$, a germ g at $a \in V$ which vanishes on V belongs to the ideal $\mathscr{I}$ defining $\mathcal{O}_V$. The Nullstellensatz implies that $g^k \in \mathscr{I}_a$ if k is large enough. But it is then clear that $g \in \mathscr{I}_a$ if $\mathcal{O}_{V,a}/\mathscr{I}_a$ is free from nilpolent elements.

Given an analytic space $(X, \mathcal{O}_X)$ we can associate to it a reduced space in the following way. Let $\mathscr{N}_x$ be the ideal in $\mathcal{O}_{X,x}$ consisting of all nilpotent elements (the nil-radical of 0). Then $\mathscr{N} = U\mathscr{N}_x$ is a coherent sheaf by the Oka-Cartan theorem, for in a local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$ we have $\mathscr{N}_X = (\mathscr{I}'/\mathscr{I})_X$ where $\mathscr{I}'$ is the sheaf of germs vanishing on V and $\mathscr{I}$ the sheaf of ideals defining $\mathcal{O}_V$. The sheaf $\mathcal{I}'$ is coherent by the Oka-Cartan then, and $\mathcal{I}$ by assumption, hence $\mathcal{I}'/\mathcal{I}$ is coherent. Now define $(X_{red}, \mathcal{O}_{X red})$ by taking X_{red} equal to X as a topological space, and $\mathcal{O}_{X red} = \mathcal{O}_X/\mathcal{N}$.

For a systematic treatment of reduced analytic spaces we refer to Narasimhan [9]. We remark here that for non-reduced spaces, the decomposition into irreducible components has no meaning, even at a point.

Example. Consider the analytic subspace X of $\mathbf{C}^2$ defined by the ideal $\mathcal{I}$ generated by $x_1 x_2$ and x_2^2. It is clear that $\mathcal{I}_X = (x_2)$ if $x_1, \neq 0$, hence X is locally the one-dimensional manifold $x_2 = 0$ outside the origin. However, $\mathcal{I} = (x_2) \cap (x_1, x_2^2)$ which is strictly contained in (x_2) at the origin so the origin cannot be an ordinary point, in particular X is not an analytic subspace of the manifold $x_2 = 0$. To illustrate this further, let $\pi : X \to \mathbf{C}$ be the projection of X into $\mathbf{C}$ defined by $(x_1, x_2) \to x_1$. We shall calculate the fibers $\pi^{-1}(a) = X \times_{\mathbf{C}} \{a\}$ of this map for an arbitrary point $a \in \mathbf{C}$.

To do this, we use the characterisation of $\mathcal{O}_{\pi^{-1}(a),b}$ given in § 1.3, example 4: if $a (= x_1) \neq 0$, and $b = (a,0)$ we find immediately $\mathcal{O}_{\pi^{-1}(a),b} = \mathbf{C}$ hence $\pi^{-1}(a)$ is a simple point. But, if $a = 0$, $b = (0,0)$ we find $\mathcal{O}_{\pi^{-1}(a),b} = \mathbf{C}\{x_1, x_2\}/(x_1, x_2^2) \simeq \mathbf{C}\{x_2\}/(x_2^2)$; hence $\pi^{-1}(0)$ is a double point.

Chapter 2.

DIFFERENTIAL CALCULUS ON ANALYTIC SPACES

Very little is known yet about differential operators on spaces with singularities. We shall just give the main definitions here. Let us first consider differential operators in the regular case, i.e. on manifolds. One then usually introduces, for each point a on a complex manifold X, the vector space $\mathcal{O}_{X,a}/\mathfrak{m}_a^{k+1}$, the jets of order k at a. Here $\mathfrak{m}_a$ denotes, as usual, the maximal ideal in $\mathcal{O}_{X,a}$. The jets of order k form, in a natural way, an analytic bundle J^k. A differential operator is then by definition a morphism of J^k into the trivial bundle $X \times \mathbf{C}$. Differential operators from bundles to bundles are defined similarly.

This definition is not suitable for generalization to analytic spaces (the collection of vector spaces $\mathcal{O}_{X,a}/\mathfrak{m}_a^{k+1}$ would not define a bundle over X). However, as noted by Grothendieck [4], if we consider, instead of the bundle J^k, the sheaf of sections of it, we can generalize to any analytic space X the definition above in the following way: