

## 3.2. Germs of analytic spaces.

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*Remarks.* 1. The same proof applies to the real case, and, more generally, to analytic algebras over a complete valued field.

2. In the  $C^\infty$  case (over  $\mathbf{R}$ ), it is known that the existence part of theorem 3.1.1. is true. Therefore steps 1 and 2 of the preceding proof are applicable, but not step 3 (the lifting  $\tilde{f}$  cannot be constructed a priori, so one has to suppose that such a lifting exists).

### 3.2. Germs of analytic spaces.

This concept will be introduced in terms of categories. As objects, we take triples  $(X, \mathcal{O}_X, x)$  where  $(X, \mathcal{O}_X)$  is an analytic space, and  $x$  a point of  $X$ ; as morphisms of  $(X, \mathcal{O}_X, x)$  into  $(Y, \mathcal{O}_Y, y)$  we take the germs at  $x$  of morphisms of  $(X, \mathcal{O}_X)$  into  $(Y, \mathcal{O}_Y)$ , which map  $x$  into  $y$ . To simplify the notations, we write  $(X, x)$  for  $(X, \mathcal{O}_X, x)$ .

We shall prove some results on the correspondence between analytic rings and germs of analytic spaces.

*Proposition 3.2.1.* To any germ  $(X, x)$  of an analytic space is associated an analytic ring  $\mathcal{O}_{X,x}$ . Every analytic ring is obtained in this way. Every morphism  $(X, x) \rightarrow (Y, y)$  of germs of analytic spaces induces a homomorphism  $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  of analytic rings. Conversely every homomorphism  $B \rightarrow A$  of analytic rings is obtained from a morphism of corresponding germs of analytic spaces; the latter is unique.

*Proof.* If  $(X, x)$  is a germ of analytic spaces,  $\mathcal{O}_{X,x}$  is an analytic ring by definition. Now let  $A = \mathbf{C} \{x_1, \dots, x_n\}/I$  be an analytic ring. We choose generators  $f_1, \dots, f_p$  for  $I$  and take an open neighborhood  $U$  of 0 such that representatives of  $f_1, \dots, f_p$  which are analytic in  $U$  can be found. These generators then define a coherent sheaf  $\mathcal{I}$  of ideals on  $U$  which defines an analytic subspace  $X$  of  $U$  with  $\mathcal{O}_{X,0} = A$ .

If  $f : B \rightarrow A$  is a homomorphism of analytic rings, we shall construct a morphism  $(X, 0) \rightarrow (Y, 0)$  of corresponding germs which induces  $F$ . We may suppose

$$A = \mathbf{C} \{x_1, \dots, x_n\}/(f_1, \dots, f_p), \quad B = \mathbf{C} \{y_1, \dots, y_m\}/(g_1, \dots, g_q);$$

as we have seen in § 1,  $F$  can be lifted into a homomorphism  $F^1 : \mathbf{C} \{y_1, \dots, y_m\} \rightarrow \mathbf{C} \{x_1, \dots, x_n\}$ ; we can choose 1) open sets  $U \subset \mathbf{C}^n$ ,  $V \subset \mathbf{C}^m$  with  $0 \in U$ ,  $0 \in V$  2) holomorphic functions  $\bar{f}_1, \dots, \bar{f}_p$  in  $U$  and  $\bar{g}_1, \dots, \bar{g}_q$  in  $V$  such that their germs at 0 are precisely the  $f_i$ 's and the  $g_j$ 's, and 3) an holomorphic mapping  $\Phi : U \rightarrow V$ , with  $\Phi(0) = 0$  such that  $\Phi^*$  induces  $F'$  at the origin.

Denote now by  $\mathcal{I}$  (resp  $\mathcal{J}$ ) the coherent sheaf of ideals generated in  $U$  (resp.  $V$ ) by the  $\bar{f}_i$ 's | resp. the  $\bar{g}_j$ 's). We have  $\Phi^*(\mathcal{J})_0 \subset \mathcal{I}_0$ , hence, since  $\mathcal{J}$  is finitely generated by restricting  $U$  and  $V$  if necessary, we have  $\Phi^*(\mathcal{J}) \subset \mathcal{I}$ . Finally we take  $X = \text{supp } \mathcal{O}_U/\mathcal{I}$ ,  $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I} |_X$  and the same for  $Y$ ; it is clear that  $\Phi$  induces the required morphism  $(Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ .

Finally, if two morphisms  $\varphi, \psi : (X, 0) \rightarrow (Y, 0)$  induce the same homomorphism  $\mathcal{O}_{Y,0} \rightarrow \mathcal{O}_{X,0}$ , we have to prove that  $\varphi$  and  $\psi$  are equals. We may assume that  $Y$  is given by a local model  $(Y, \mathcal{O}_V | \mathcal{J} | Y)$  for some coherent sheaf  $\mathcal{J}$  of ideals on an open set  $V \subset \mathbf{C}^m$ ; by composition with the injection  $Y \rightarrow V$ , we may restrict ourselves to the case where  $Y = \mathbf{C}^m$ ; the morphisms  $\varphi$  and  $\psi$  are now given by sections  $f, g \in \Gamma(X, \mathcal{O}_X^m)$ , and the hypothesis means that the germs of  $f$  and  $g$  at  $0$  coincide; hence  $f$  and  $g$  coincide in a neighborhood of  $0$  in  $X$ , which proves the assertion.

### 3.3 Finite morphisms

Let  $f : (X, 0) \rightarrow (Y, 0)$  be a morphism of germs of analytic spaces. Then  $f$  is called "finite" if the corresponding homomorphism  $f^* : \mathcal{O}_{Y,0} \rightarrow \mathcal{O}_{X,0}$  makes  $\mathcal{O}_{X,0}$  finite over  $\mathcal{O}_{Y,0}$ . According to the preparation theorem 3.1.3. in order that  $f$  be finite, it is necessary and sufficient that  $\mathcal{O}_{X,0}/\mathfrak{M}(\mathcal{O}_{Y,0}) \mathcal{O}_{X,0}$  be finite over  $\mathbf{C}$ ; in geometrical terms, this means that the germ of space  $f^{-1}(0)$  is finite over the point  $0$  (see § 1.3, example 4).

In the global case (complex or real), we give the following definition:

*Definition 3.3.1.* A morphism of separated analytic spaces  $f = (f_0, f^1) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is finite if the following properties hold:

- 1)  $f$  is proper (i.e.  $f_0$  is proper).
- 2) For any point  $x \in X$ , the induced morphism of germs  $f_x : (X, \mathcal{O}_X, x) \rightarrow (Y, \mathcal{O}_Y, f_0(x))$  is finite.

In the *complex* case, we have the following results :

*Proposition 3.3.2.*  $f$  is finite if and only if  $f$  is proper and, for any  $b \in Y$ , the set  $f_0^{-1}(b)$  is finite.

This proposition is more or less equivalent to the "Nullstellensatz"; for the proof see e.g. Houzel [6] or Narasimhan [9]. In the real case, the part "if" of this proposition is not even true when  $Y$  is a point : for instance the subspace of  $\mathbf{R}^2$  defined by  $\mathcal{I} =$  (coherent sheaf of ideals generated by  $x_1^2 + x_2^2$ ) has support  $0$ ; but  $\mathbf{R}\{x_1, x_2\}/(x_1^2 + x_2^2)$  is not finite over  $\mathbf{R}$ .

*Proposition 3.3.2.* If  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a finite morphism, then the direct image  $f_* (\mathcal{O}_X)$  is a coherent analytic sheaf of  $\mathcal{O}_Y$ -modules; converse-