

4.1. Stein spaces

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

ly, let \mathcal{A} be a sheaf of \mathcal{O}_Y -algebras, which is coherent as sheaf of \mathcal{O}_Y -modules. Then there exists an analytic space (X, \mathcal{O}_X) and a finite morphism $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ such that $f_*(\mathcal{O}_X)$ is isomorphic with \mathcal{A} as sheaf of \mathcal{O}_Y -algebras ; the triple (X, \mathcal{O}_X, f) is unique up to an isomorphism.

We do not prove this proposition here and refer to Houze! [6] or Narasimhan [9] for this proof. We note also that a proof of the direct part can be given along the same lines as theorem 3.1.3, combined with the fact that direct images under finite morphism preserve exact sequences of sheaves of \mathcal{O}_X -modules (in other words, that higher direct images are zero). We note also that, for *proper* morphisms (not necessarily finite), a much deeper result has been proved by Grauert [2], [3].

Finally, we remark that, in the real case, proposition 3.3.2. is false (take, for instance, X the submanifold of \mathbf{R}^2 defined by $x_2 - x_1^2 = 0$, $Y = \mathbf{R}$ and $f =$ the projection on the x_2 -axis ; $f_*(\mathcal{O}_X)$ has support $x_2 \geq 0$, which is not an analytic subset of \mathbf{R} , hence $f_*(\mathcal{O}_X)$ cannot be coherent!)

CHAPTER 4.

THE FINITENESS THEOREM

In this chapter, we consider only *complex* analytic spaces, separated and having a countable basis of open sets.

4.1. Stein spaces

Let (X, \mathcal{O}_X) be an analytic space, and K a subset of X ; we denote, as usual by \hat{K} the set

$$\left\{ x \in X \mid \forall f \in \Gamma(X, \mathcal{O}_x) : |f(x)| \leq \sup_{y \in K} |f(y)| \right\}$$

Definition 4.1.1. a) (X, \mathcal{O}_X) is called holomorphically convex if, for any K compact $\subset X$, \hat{K} is compact ;
 b) (X, \mathcal{O}_X) is called a Stein space if it is holomorphically convex, and if, for any $x \in X$, there exist sections $f_1, \dots, f_p \in \Gamma(X, \mathcal{O}_X)$ with $f_i(x) = 0$, such that x is an isolated point of the counter-image of 0 in the morphism $(X, \mathcal{O}_X) \rightarrow \mathbf{C}^p$ defined by f_1, \dots, f_p , (This last property can also be expressed as the fact that the morphism of germs : $(X, \mathcal{O}_X, x) \rightarrow (\mathbf{C}^p, 0)$ defined by f_1, \dots, f_p is finite).

If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (“Theorems A and B ” of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space (X, \mathcal{O}_X) . Then

- 1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
- 2) For $p \geq 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let (X, \mathcal{O}_X) be a closed analytic subspace of a domain of holomorphy $U \subset \mathbb{C}^n$; if F is an analytic coherent sheaf on X , let \tilde{F} be the trivial extension of F to U ; then \tilde{F} is a coherent sheaf of \mathcal{O}_U modules, and theorems A and B are valid for \tilde{F} : therefore, they are true for F .

4.2. Topology on $\Gamma(X, F)$.

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbb{C}^n$; and, with the previous notations, suppose that \tilde{F} admits a *finite presentation* i.e. an exact sequence of sheaves of \mathcal{O}_U -modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \rightarrow 0.$$

Applying theorem B to the exact sequences

$$0 \rightarrow \text{Im } \alpha \rightarrow \mathcal{O}_U^p \rightarrow \tilde{F} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Ker } \alpha \rightarrow \mathcal{O}_U^q \rightarrow \text{Im } \alpha \rightarrow 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U, \alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U, \beta)} \Gamma(U, \tilde{F}) \rightarrow 0.$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\text{Im } \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\text{Im } \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \text{Im } (\alpha_x)$, hence $f \in \Gamma(U, \text{Im } \alpha)$; but, according to theorem B , the mapping $\Gamma(U, \mathcal{O}_U)^q \rightarrow \Gamma(U, \text{Im } \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \text{Im } \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of \tilde{F} (in fact, it does not even depend on the imbedding $X \rightarrow U$, but we shall not need it here). For, suppose we have a second presentation