

Zeitschrift: L'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ANALYTIC SPACES
Kapitel: 4.2. Topology on (X, F) .
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DOI: <https://doi.org/10.5169/seals-42341>

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If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (“Theorems A and B ” of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space (X, \mathcal{O}_X) . Then

- 1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
- 2) For $p \geq 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let (X, \mathcal{O}_X) be a closed analytic subspace of a domain of holomorphy $U \subset \mathbb{C}^n$; if F is an analytic coherent sheaf on X , let \tilde{F} be the trivial extension of F to U ; then \tilde{F} is a coherent sheaf of \mathcal{O}_U modules, and theorems A and B are valid for \tilde{F} : therefore, they are true for F .

4.2. Topology on $\Gamma(X, F)$.

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbb{C}^n$; and, with the previous notations, suppose that \tilde{F} admits a *finite presentation* i.e. an exact sequence of sheaves of \mathcal{O}_U -modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \rightarrow 0.$$

Applying theorem B to the exact sequences

$$0 \rightarrow \text{Im } \alpha \rightarrow \mathcal{O}_U^p \rightarrow \tilde{F} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Ker } \alpha \rightarrow \mathcal{O}_U^q \rightarrow \text{Im } \alpha \rightarrow 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U, \alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U, \beta)} \Gamma(U, \tilde{F}) \rightarrow 0.$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\text{Im } \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\text{Im } \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \text{Im } (\alpha_x)$, hence $f \in \Gamma(U, \text{Im } \alpha)$; but, according to theorem B , the mapping $\Gamma(U, \mathcal{O}_U)^q \rightarrow \Gamma(U, \text{Im } \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \text{Im } \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of \tilde{F} (in fact, it does not even depend on the imbedding $X \rightarrow U$, but we shall not need it here). For, suppose we have a second presentation

$$\Gamma(U, \mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U, \mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \rightarrow 0.$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$ -linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta = \beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p / \text{Im } \Gamma(U, \alpha) \rightarrow \Gamma(U, \mathcal{O}_U)^{p'} / \text{Im } \Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. General case

If X is an analytic space and F an analytic coherent sheaf on X , we can find a) a locally finite covering of X by open subspaces X_i , b) for each i , a morphism $X_i \rightarrow U_i$, U_i open polycylinder in \mathbf{C}^{n_i} , which identifies X_i with a closed subspace of U_i c) for each i , a coherent sheaf \tilde{F}_i on U_i admitting a finite presentation, such that \tilde{F}_i is the extension of $F|_{X_i}$.

On $\Gamma(X_i, F|_{X_i})$ we have already defined a topology; further, consider the natural injection

$$\Gamma(X, F) \rightarrow \prod_i \Gamma(X_i, F|_{X_i})$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j (= X_i \times_X X_j)$, we have $(f_i)_x = (f_j)_x$; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has two coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X , the restriction map $\Gamma(X, F) \rightarrow \Gamma(X', F|_{X'})$ is continuous. If X' is relatively compact in X , then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j , there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i , and applying Ascoli's theorem).

4.3. Topology on $H^p(X, F)$

We consider a locally finite covering $\mathcal{U} = \{X_i\}_{i \in I}$ by open subspaces of the preceding type. If we have $i_0, \dots, i_p \in I$, we consider the natural morphisms

$$X_{i_0 \dots i_p} = X_{i_0} \times_X \dots \times_X X_{i_p} \rightarrow X_{i_0} \times \dots \times X_{i_p} \rightarrow U_{i_0} \times \dots \times U_{i_p}$$