§4. Algebraic study of flatness

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The set of " nice " fibers is dense in X, so we cannot remove the z-axis and still get a closed subspace of $\mathbf{C}_3$.

§ 4. *Algebraic study of flatness*

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A-module E is *flat*, if for every exact sequence of A-modules

$$0 \to F' \to F \to F'' \to 0\,,$$

the sequence $0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is also exact. We can also say, because $\otimes$ is right exact, that E is flat, if for every injective homomorphism $F' \to F$, $E \otimes F' \to E \otimes F$ is also injective.

Examples of modules which are not flat:

(1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\,\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \to \mathbf{Z} \overset{2I}{\to} \mathbf{Z}$ $(2I : x' \to 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \overset{2I}{\to} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So $\mathbf{Z}_2$ is not a flat $\mathbf{Z}$ module.

(2) If $A = \mathbf{C}\,\{x\}$, $E = \mathbf{C} = \mathbf{C}\,\{x\}/(x)$, $F = F' = \mathbf{C}\,\{x\}$, then the sequence $0 \to F \overset{xI}{\to} F'$ $(xI : p\,(x) \to xp\,(x))$ is exact. But the homomorphism $E \overset{xI}{\to} E$ is not injective.

Proposition 1: If A is an integral domain and E a flat A-module, then E is torsion-free.

Proof: Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \to A A \overset{aI}{\to}$ $(aI : x \to ax)$ is exact. Since E is flat, the sequence $0 \to E \overset{aI}{\to} E$ is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

Examples of flat modules:

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if E is free and finite type, then $E = A^n$ and $E \otimes F = F^n$. If $F' \to F$ is injective, so is $F'^n \to F^n$ too.

If E is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of E follows from (1).

(3) Let S be a multiplicative system in A. Then the ring of fractions $S^{-1} A$ is a flat A-module. In fact the ring $S^{-1} A$ can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that S has only regular elements. We can define in the set S a partial order in the following way:

$$s' \geqq s \Leftrightarrow \exists\, t \in A\,, \quad ts = s' \quad \text{(such a } t \text{ is then unique).}$$

Let $E_s = A$ for every $s \in S$, and if $s' \geqq s$ (i.e. $s' = ts$) then let $f_s^{s'}$ be the homomorphism $t.\ I_A : E_s \to E_{s'}$. The family $(E_s)_{s \in S}$ with the homomorphisms $(f_s^{s'})$ is an inductive system.

Let $E = \lim\limits_{\to} E_s$ be the inductive limit of this system, and φ_s the canonical homomorphism $E_s \to E$. We shall define an isomorphism $\psi : E \to S^{-1} A$.

We first define for every s a homomorphism $\psi_s : E_s = A \to S^{-1} A$; $x \to x/s$. Now if $s' \geqq s$, then

$$(\psi_s{}' \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x)\,.$$

Therefore there exists a homomorphism $\psi : E \to S^{-1} A$, satisfying $\psi_s = \psi \circ \varphi_s$ for every $s \in S$.

Because every element of $S^{-1} A$ has the form a/s, ψ is surjective. On the other hand if $\psi(\phi_s(x)) = 0$, then $\psi_s(x) = x/s = 0$. Thus $x = 0$, and ψ is also injective.

The above proof can be extended to the general case, not assuming that the elements of S are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

Theorem 1: (Daniel, Lazard)

Any flat module is a inductive limit of free modules.

For the proof: See *C.R. Acad. Sci. Paris,* 258 (1964), pp. 6313-6316.

Some elementary properties of flat modules:

(1) If E and F are flat A-modules, then $E \underset{A}{\otimes} F$ is also flat. In fact, if $G' \to G$ is injective, then $F \otimes G' \to F \otimes G$ is injective, and also $E \otimes (F \otimes G') \to$ $\to E \otimes (F \otimes G)$ is injective. The result follows from the assosiativity of the tensor product.

(2) Let $\phi : A \to B$ be a ring homomorphism, and E a flat A-module. The module $B \underset{A}{\otimes} E$ is a flat B-module.

If F is a B-module, then $F \underset{B}{\otimes} (B \underset{A}{\otimes} E) = (F \underset{B}{\otimes} B) \underset{A}{\otimes} E = F \underset{A}{\otimes} E$ further if F' and F are B-modules, and $F' \to F$ an injective homomorphism of B-modules, we can consider this homomorphism as an injective homomorphism of A-modules. Because E is A-flat,

$$F' \otimes_A E \to F \otimes_A E \text{ is injective.}$$

(3) Let $\phi : A \to B$ be a ring homomorphism, such that B is a flat A-module. If F is a flat B-module, then F is a flat A-module. In fact: if $E' \to E$ is injective, then $E' \underset{A}{\otimes} B \to E \underset{B}{\otimes} B$ is injective, and also $(E \underset{A}{\otimes} B) \underset{B}{\otimes} F' \to (E \underset{A}{\otimes} B) \underset{B}{\otimes} F$ is injective. But $(E' \otimes_A B)_B \otimes F' = E' \otimes_A F$; $(E \otimes_A B) \otimes_B F = E \otimes_A F$.

If an A-module E is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor.

Definition 2: A free resolution of E is an exact sequence: $\ldots \to L_n \to L_{n-1} \to$ $\to \ldots \to L_1 \to L_0 \to E \to 0$, where all L_i are free A-modules.

The complex of the resolution is the sequence

$$\text{(L.)} \ldots \to L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to 0 .$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensorproducts $L_i \otimes F$, we get

$$(\text{L.} \otimes F) \ldots \to L_n \otimes F \to L_{n-1} \otimes F \to \ldots \to L_1 \otimes F \to L_0 \otimes F \to 0 .$$

Definition 3:

$$\text{Tor}_n^A (E, \text{F}) = H_n (\text{L.} \otimes F) = \frac{\text{Ker}(L_n \otimes F \to L_{n-1} \otimes F)}{\text{Im}\,(L_{n+1} \otimes F \to L_n \otimes F)}$$

if $n \geqq 1$, and $\text{Tor}_0^A (E, F) = \text{Coker}\,(L_1 \otimes F \to L_0 \otimes F) = E \otimes F$.

Basic properties of Tor:

(1) $\text{Tor}_n (E, F)$ is independent of the choice of the resolution (up to a canonical isomorphism).

(2) If we take a free resolution of F, we get $\operatorname{Tor}_n(F, E) = \operatorname{Tor}_n(E, F)$ (Symmetry of the Tor). We can also define $\operatorname{Tor}_n(E, F)$ by taking two free resolutions, one of E and one of F.

(3) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence, then we get a long exact sequence:

$$\begin{aligned} & \operatorname{Tor}_n(E', F) \to \operatorname{Tor}_n(E, F) \to \operatorname{Tor}_n(E'', F) \to \\ \to\ & \operatorname{Tor}_{n-1}(E', F) \to \operatorname{Tor}_{n-1}(E, F) \to \operatorname{Tor}_{n-1}(E'', F) \to \\ \to\ & \text{— — — — — — — — — — — — — — — —} \to \\ \to\ & \operatorname{Tor}_1(E', F) \to \operatorname{Tor}_1(E, F) \to \operatorname{Tor}_1(E'', F) \to \\ \to\ & E' \otimes F \to E \otimes F \to E'' \otimes F \to 0 . \end{aligned}$$

(4) Tor is compatible with inductive limit, i.e. if $E = \varinjlim (E_i)$, then

$\operatorname{Tor}_n(\varinjlim E_i, F) = \varinjlim (\operatorname{Tor}_n(E_i, F))$.

(5) We can define $\operatorname{Tor}_n(E, F)$ by taking a flat resolution of E.

Proposition 3: Let E be an A-module. Then the following conditions are equivalent:

(a) E is flat.

(b) For all A-modules F, and for all $n \geqq 1$, $\operatorname{Tor}_n(E, F) = 0$.

(c) For all A-modules F, $\operatorname{Tor}_1(E, F) = 0$.

Proof: (a) ⇒ (b). If $\dots \to L_n \to L_{n-1} \to \dots \to L_1 \to L_0 \to F \to 0$ is a free resolution of F, then the sequence

$$\dots \to E \otimes L_n \to E \otimes L_{n-1} \to \dots \to E \otimes L_1 \to E \otimes L_0 \to E \otimes F \to 0$$

is exact, thus $\operatorname{Tor}_n(E, F) = 0$ for all $n \geqq 1$.

(b) ⇒ (c) clear. (c) ⇒ (a): If the sequence $0 \to F' \to F \to F'' \to 0$ is exact, so is also (by (3) above) $\operatorname{Tor}_1(E, F'') \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$. Now $\operatorname{Tor}_1(E, F'') = 0$, thus E is flat.

Proposition 4: If I and J are two ideals in A, then $\operatorname{Tor}_1^A(A/I, A/J) = I \cap J / I . J$.

Proof: From the exact sequence $0 \to I \to A \to A/I \to 0$, we get the exact sequence:

$$\operatorname{Tor}_1(A, A/J) \to \operatorname{Tor}_1(A/I, A/J) \to I \otimes A/J \to A \otimes A/J \to A/I \otimes A/J \to 0 .$$

But now $\operatorname{Tor}_1(A, A/J) = 0$ (A beeing A-free), and $I \otimes A/J = I/I . J$; $A \otimes A/J = A/J$. Therefore the sequence $0 \to \operatorname{Tor}_1(A/I, A/J) \to I/I . J \to A/J$ is exact, and $\operatorname{Tor}_1(A/I, A/J) = \operatorname{Ker}(I/I . J \to A/J) = I \cap J/I . J$.

Example : Let U be an open set in $\mathbf{C}^n$, and $x \in U$. Further let $X, Y \subset U$ be two hypersurfaces, defined by $I = (f)$ and $J = (g)$. Supposing that f and g do not have common factors: $I_x \cap J_x = I_x J_x$, and

$$\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = \operatorname{Tor}_1(\mathcal{O}_{U,x}/I_x, \quad \mathcal{O}_{U,x}/J_x) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0.$$

Heuristic remark : The formula $\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ expresses the fact that X and Y are " in general position ". If for example X and Y are two linears subspaces in $\mathbf{C}^n$ of dimensions p and q, we have $\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ if $\dim(X \cap Y) = p+q-n$, and $\operatorname{Tor}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$ otherwise.

Next we shall prove an elementary flatness criterion.

Proposition 5 : Let E be an A-module. The following conditions are equivalent:

(*a*) E is flat.

(*b*) For all finitely generated ideals I of A, $\operatorname{Tor}_1(E, A/I) = 0$.

(*c*) For all monogenous A-modules F, $\operatorname{Tor}_1(E, F) = 0$.

Proof : (*a*) $\Rightarrow$ (*b*), by prop. 3.

(*b*) $\Rightarrow$ (*c*): Because Tor is compatible with inductive limit, we can suppose, that $\operatorname{Tor}_1(E, A/I) = 0$ for an arbitrary ideal I of A. But every monogenous A-module F can be represented by A/I.

(*c*) $\Rightarrow$ (*a*). By prop. 3 it is sufficient to prove that $\operatorname{Tor}_1(E, F) = 0$ for any A-module F.

First consider the case, where F is finitely generated. We use induction, supposing that $\operatorname{Tor}_1(E, F) = 0$, when F has n generators. Let F have $(n+1)$ generators $x_1, \ldots, x_n, x_{n+1}$. If F' is the submodule generated by $\{x_1, \ldots, x_n\}$, then $F' \subset F$ and $F'' = F/F'$ is monogenous. The exact sequence $0 \to F' \to F \to F'' \to 0$ gives the exact sequence $\operatorname{Tor}_1(E, F') \to \operatorname{Tor}_1(E, F) \to \operatorname{Tor}_1(E, F'')$. Now $\operatorname{Tor}_1(E, F') = \operatorname{Tor}_1(E, F'') = 0$, thus $\operatorname{Tor}_1(E, F) = 0$. In the general case, F can be considered as an inductive limit of finitely generated modules, and because Tor is compatible with inductive limits, $\operatorname{Tor}_1(E, F) = 0$.

Proposition 6 : Let A be an integral domain, and E an A-module. Then E is torsionfree if and only if $\operatorname{Tor}_1(E, A/(a)) = 0$, for any element $a \in A$.

Proof : If E is A-module, $a \in A$, then the exact sequence $0 \to A \xrightarrow{aI} A \to A/(a) \to 0$ gives the exact sequence $0 \to \operatorname{Tor}_1(E, A/(a)) \to E \xrightarrow{aI} E$. In other words $\operatorname{Tor}_1(E, A/(a)) = \{x \in E \mid ax = 0\}$, from which the result follows.

Corollary: Let A be a principal ideal domain. E is flat if and only if E is torsionfree.

Proof: We have already proved that, if E is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

Theorem 2: Let A be a noetherian local ring with maximal ideal m; $k = A/m$, and E a finitely generated A-module. The following conditions are equivalent:

(*a*) E is free.

(*b*) E is flat.

(*c*) $\mathrm{Tor}_1^A(E, k) = 0$.

Proof: We have already proved $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$: We recall first Nakayma's lemma. If A is a local ring with maximal ideal m; $k{=}A/m$, and E is a finitely generated A-module, such that $k \underset{A}{\otimes} E = E/mE = 0$, then $E = 0$.

The module $\bar{E} = k \underset{A}{\otimes} E = E/mE$ is a finitely generated vector space over k. Let $\{\bar{x}_1. \ldots, \bar{x}_r\}$ be a base of $\bar{E}$ (over k), and $\{x_1, \ldots, x_r\}$ E representatives of $\bar{x}_i : s$. Consider the homomorphism $\phi : A^r \to E$, $\phi(a_1, \ldots, a_r) = \sum a_i x_i$. Denoting by R and Q the kernel and the cokernel of ϕ, we get an exact sequence:

$$(*) \qquad 0 \to R \to A^r \to E \to Q \to 0$$

and R, Q are finitely generated A-modules. From (*) we get the exact sequence

$$A^r \underset{A}{\otimes} k \to E \underset{A}{\otimes} k \to Q \underset{A}{\otimes} k \to 0\,.$$

But $\bar{E} = E \underset{A}{\otimes} k \simeq k^r = A^r \underset{A}{\otimes} k$, so $Q \underset{A}{\otimes} k = 0$, and by Nakayama's lemma $Q = 0$.

Therefore ge have an exact sequence

$$0 \to R \to A^r \to E \to 0\,.$$

From this we get: $\mathrm{Tor}_1(E, k) \to k \underset{A}{\otimes} R \to k^r \to \bar{E} \to 0$ (exact). Now: $\bar{E} \simeq k^r$, $\mathrm{Tor}_1(E, k) = 0$ (by assumption). Therefore $k \underset{A}{\otimes} R = 0$, and once more by Nakayama's lemma $R = 0$, thus $E \simeq A^r$, i.e. E is free.

Proposition 7: Let $\phi : A \to B$ be a ring homomorphism, and let B be A-flat. If I is an ideal of A, we write $\bar{A} = A/I$, $\bar{B} = B/IB = \bar{A} \underset{A}{\otimes} B$. Let F be a B-module, then: $\mathrm{Tor}_i^A(\bar{A}, F) = \mathrm{Tor}_i^B(\bar{B}, F)$ $(i \geqq 0)$.

Proof: We choose first a B-free resolution of F

$$\to L_{n+1} \to L_n \to \ldots \to L_1 \to L_0 \to F \to 0 .$$

If $L.$ is the respective complex of resolution, then

$$\bar{B} \underset{B}{\otimes} L. = B/IB \underset{B}{\otimes} L. = \bar{A} \underset{A}{\otimes} (B \underset{B}{\otimes} L.) = \bar{A} \underset{A}{\otimes} L.$$

Because every L_i is B-free, and B is A-flat, every L_i is A-flat (Property 3 after Th. 1). Thus $L.$ is a flat A-resolution, and

$$\mathrm{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \underset{A}{\otimes} L.) = H_i(\bar{B} \underset{B}{\otimes} L.) = \mathrm{Tor}_i^B(\bar{B}, F) .$$

We shall next state the second flatness criterion for noetherian local rings.

Theorem 3: Let A and B be two noetherian local rings, with maximal ideals $\underline{m}$, $\underline{n}$; $k = A/\underline{m}$. If $\phi : A \to B$ is a local homomorphism (i.e. $\phi(\underline{m}) \subset \underline{n}$), and F finitely generated B module then

$$F \text{ is A-flat} \Leftrightarrow \mathrm{Tor}_1^A(k, F) = 0 .$$

The proof of this theorem is much more difficult than that of th. 20 see for example:
Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, $(i) \Leftrightarrow (iii)$, p. 98.

The conditions in Bourbaki's theorem are here fullfilled:

1° A finitely generated module F over a noetherian local ring B is idealwise separated for $\underline{n}$. (*Ibid.*, § 5. 1. Ex. 1, p. 97.)

2° If $\phi : A \to B$ is a local homomorphism, F is also idealwise separated for $\underline{m}$. (*Ibid.*, § 5, prop. 2, p. 101.)

3° Also the flatness condition is fulfilled, because k is a field.

Remark: The main interest of the theorem lies in the fact, that it is true without any assumption of finitness on B.

Corollary: If the assumptions are the same as in the theorem 3, and if moreover B is A-flat, then

$$F \text{ is } A\text{-flat} \Leftrightarrow \mathrm{Tor}_1^B(\bar{B}, F) = o ,$$

where $\bar{B} = B/\underline{m}B$.

Proof: $\mathrm{Tor}_1^A(k, F) = \mathrm{Tor}_1^B(\bar{B}, F)$, by prop. 7.

§ 5. *Geometric applications of the flatness criterions*

A) *Flatness for finite morphisms*

Proposition 1: Let $\pi\colon X \to S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_*(\mathcal{O}_X)$ is a coherent analytic sheaf over S. The following conditions are equivalent:

(*a*) π is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, $s = \pi(x)$).

(*b*) For every s, $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$-module.

(*c*) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

Proof: Because π is finite $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \varepsilon \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is $(b) \Rightarrow (c)$.

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and $\mathcal{E}$ a coherent $\mathcal{O}_s$-module. Let $E(s)$ be the finite dimensional vector space (over $\mathbf{C}$) $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s$. $\mathcal{E}$ is a locally free $\mathcal{O}_{S,s}$-module if an only if $\dim_{\mathbf{C}} E(s)$ is locally constant.

Proof: If $\mathcal{E}$ is locally free, then $\dim_{\mathbf{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbf{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \to \mathcal{E}_U \to 0$ is exact. d is determined by a $p \times q$ matrix of analytic functions on U, so it gives a morphism $\mathbf{C}_U^p \xrightarrow{d} \mathbf{C}_U^q$ of trivial vector bundles over U.

From the exact sequence $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \to \mathcal{E}_s \to 0$, we get (by making tensor-products with $\mathbf{C}_s$) the exact sequence:

$$\mathbf{C}_s^p \xrightarrow{d(s)} \mathbf{C}_s^q \to E(s) \to 0\,,$$

which shows that d has constant rank in U. Thus Ker d and Im d are vector bundles, and we can write

$$\mathbf{C}_U^p = F_1 \oplus G_1\,, \quad \mathbf{C}_U^q = F_0 \oplus G_0\,,$$

$$d : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_0\,. \end{cases}$$