

## §2. Privileged polycylinders

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$$g|_W : \begin{cases} F_2 & \rightarrow 0 \\ G_2 W & \simeq F_3 \end{cases} .$$

If  $p: E_{2W} \rightarrow F_2$  is the projection with kernel  $G_{2W}$ , the map,  $p \circ f: E_{1W} \rightarrow F_2$  is a split epimorphism in  $x_0$ . Again by prop. 2 we have over an open neighbourhood  $U \subset W$  of  $x_0$  a decomposition  $E_{1U} = F_1 \oplus G_{1U}$  (with  $F_1 = \text{Ker } p \circ f$ )

$$(p \circ f)|_U : \begin{cases} F_1 & \rightarrow 0 \\ G_{1U} & \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image  $f|_U(F_1)$  is contained in  $G_{2U}$ . But  $g|_U \circ f|_U = 0$  and  $g|_{G_{2U}}$  is a monomorphism hence  $f|_U: F_1 \rightarrow 0$ . We get finally (restricting all our morphisms to  $U$ )

$$f|_U : \begin{cases} F_{1U} & \rightarrow 0 \\ G_{1U} & \simeq F_{2U} \end{cases} \qquad g|_U : \begin{cases} F_{2U} & \rightarrow 0 \\ G_{2U} & \xrightarrow{\sim} F_{3U} \end{cases} .$$

## § 2. Privileged polycylinders

*Definition 1:* A polycylinder in  $\mathbf{C}^n$  is a compact set  $K$  of the form  $K = K_1 \times \dots \times K_n$  where each  $K_i$  is a compact, convex subset of  $\mathbf{C}$ , with nonempty interior. If each  $K_i$  is a disc, then  $K$  is a polydisc. We first recall the following theorem of Cartan.

*Theorem 1:* Let  $K$  be a polycylinder contained in an open subset  $U$  of  $\mathbf{C}^n$ . Let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ .

(A) There exists an open neighbourhood of  $K$  over which  $\mathcal{F}$  admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

(B)  $H^q(K, \mathcal{F}) = 0$  for  $q > 0$ .

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf  $\mathcal{F}$ , the sequence

$$0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0$$

is an  $\mathcal{O}_U(K)$  - free resolution of  $\mathcal{F}(K)$ .

2) Given a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0,$$

then the sequence

$$0 \rightarrow \mathcal{F}'(K) \rightarrow \mathcal{F}(K) \rightarrow \mathcal{F}''(K) \rightarrow 0 \quad \text{is exact.}$$

Let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ , and let  $K \subset U$  be a polycylinder. If  $V$  is an open neighbourhood of  $K$ , then  $\mathcal{F}(V)$  can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give  $\mathcal{F}(K)$  the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from  $\mathcal{F}(K)$  and by choosing  $K$  in a "privileged" way.

Let  $B(K) = \{f : K \rightarrow \mathbb{C} \mid f \text{ continuous on } K \text{ and analytic on } \overset{\circ}{K}\}$ , then  $B(K)$  is Banach algebra and  $B(K) \subset C(K)$ . The sections of  $\mathcal{O}_U$  over  $K$  are elements of  $B(K)$ , and  $B(K)$  is in fact the uniform closure of  $\mathcal{O}_U(K)$  in  $C(K)$ .

If  $\mathcal{L} = \mathcal{O}_U^r$ , we define  $B(K, \mathcal{L}) = B(K)^r$ . Then  $B(K; \mathcal{L})$  is a free  $B(K)$ -module, and since  $\mathcal{L}(K) = \mathcal{O}_U(K)^r$ , we have  $B(K; \mathcal{L}) = B(K) \otimes_{\mathcal{O}_U(K)} \mathcal{L}(K)$ .

We now assume that  $\mathcal{F}$  is a coherent sheaf on  $U$ , where  $U \subset \mathbb{C}^n$  is open. Consider a free resolution

$$(R) \quad 0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 \quad \text{of } \mathcal{F}.$$

From (R) we get an  $\mathcal{O}_U(K)$ -free resolution of  $\mathcal{F}(K)$

$$(R') \quad 0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_1(K) \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0.$$

Taking the tensorproduct  $B(K) \otimes_{\mathcal{O}_U(K)}$  we get the complex

$$B(K; \mathcal{L}.): 0 \rightarrow B(K; \mathcal{L}_n) \rightarrow \dots \rightarrow B(K; \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0).$$

*Definition 2:* The polycylinder  $K$  is called  $\mathcal{F}$ -privileged if the complex  $B(K; \mathcal{L}.)$  is split-exact in every degree  $> 0$ .

*Remark:* The property of being  $\mathcal{F}$ -privileged is independent of the resolution (R).

The exactness of  $B(K; \mathcal{L}.)$  can be expressed by  $\text{Tor}_i^{\mathcal{O}_U(K)}(B(K), \mathcal{F}(K)) = 0$ , for every  $i > 0$ , and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of  $(R)$ , and this is omitted.

Since  $B(K; \mathcal{L}_i)$  is a Banach space, the image and its complement are thus Banach spaces if  $K$  is  $\mathcal{F}$ -privileged. In this case we define  $B(K; \mathcal{F}) = \text{Coker } (B(K, \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{O}_U} \mathcal{F}(K)$  and we get a  $B(K)$ -module, which is a Banach-space.

*Warning:* In the definition of split-exactness, the subspaces are splitting vector spaces, but they are not splitting  $B(K)$ -modules in general.

We have the following important theorem about the existence of privileged polycylinders:

*Theorem 2:* Let  $U$  be an open subset of  $\mathbb{C}^n$ , and let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ . For any  $x \in U$  there exists a fundamental system of neighbourhoods of  $x$  in  $U$ , which are  $\mathcal{F}$ -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

*Example:* (Curves in  $\mathbb{C}^2$ ) Let  $U \subset \mathbb{C}^2$  be an open connected neighbourhood of the origin, and let  $h: U \rightarrow \mathbb{C}$  be analytic and  $h \neq 0$ .

Let  $X$  be the curve given by  $h$ , that is  $X = h^{-1}(0)$ ,  $\mathcal{O}_X = \mathcal{O}_U / (h)$ . We have an exact sequence  $0 \rightarrow \mathcal{O}_U \xrightarrow{h} \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$ . Consider a polycylinder  $K = K_1 \times K_2 \subset U$ . By definition  $K$  is  $\mathcal{O}_X$ -privileged if and only if  $h: B(K) \rightarrow B(K)$  is a split monomorphism.

Let  $\dot{K}_j$  denote the boundary of  $K_j$ , and define  $\ddot{K} = \dot{K}_1 \times \dot{K}_2$  ( $\ddot{K}$  is called the Šilov Boundary of  $K$ ).

*Proposition 1:* (a) The following conditions are equivalent:

- (i)  $h: B(K) \rightarrow B(K)$  is a monomorphism.
- (i')  $\exists a > 0$  such that  $\|hf\| \geq a\|f\|$ ,  $\forall f \in B(K)$ .
- (ii)  $X \cap \ddot{K} = \emptyset$ .

(b) If  $(K_1 \times K_2) \cap X = \emptyset$ , then  $h$  is a split monomorphism (i.e.  $K$  is  $\mathcal{O}_X$ -privileged).

*Proof:* (a) (i)  $\Leftrightarrow$  (i') is a well known fact from the theory of normed vector spaces.

(ii)  $\Rightarrow$  (i'). Assume  $X \cap \ddot{K} = \emptyset$ . If  $f \in B(K)$ , then it follows from the maximum principle that  $\|f\| = \sup_K |f(x)| = \sup_{\ddot{K}} |f(x)|$ . Since  $h(x) \neq 0$

whenever  $x \in \overset{\circ}{K}$ , we get  $a = \inf_K |h(x)| > 0$ . Hence  $\|hf\| = \sup_K |hf(x)| \geq \geq a \sup_K |f(x)| = a \|f\|$ .

(i')  $\Rightarrow$  (ii). Suppose that  $X \cap \overset{\circ}{K} \neq \emptyset$  and  $x = (x_1, x_2) \in X \cap \overset{\circ}{K}$ . We choose an analytic function  $f_1 : U_1 \rightarrow \mathbf{C}$ , where  $U_1 \supset K_1$ , and  $U_1$  is open, such that  $f_1(x_1) = 1$ ,  $|f_1(z)| < 1$  if  $z \in K_1$ ,  $z \neq x_1$ . Similarly we choose an analytic function  $f_2 : U_2 \rightarrow \mathbf{C}$ , with the same properties. Consider the function  $f \in B(K) : (z_1, z_2) \rightarrow f_1(z_1)f_2(z_2)$ . Since  $h(x) = 0$  it follows that the sequence  $\{hf^n\}$  converges pointwise to 0 in  $K$ .

Applying Dini's theorem we get  $\|hf^n\| \rightarrow 0$ . From the inequality  $a \|f^n\| \leq \|hf^n\|$  we get  $\|f^n\| \rightarrow 0$ , which is a contradiction, because for every  $n : f^n(x) = 1$ .

(b) Use the Weierstrass preparation theorem (extended form).

*Question.* Does the condition (ii) imply that  $h : B(K) \rightarrow B(K)$  is a split monomorphism?

#### IV. FLATNESS AND PRIVILEGE

##### § 1. Morphisms from an analytic space into $B(K)$

Let  $S$  be an analytic space and  $K$  a polycylinder in an open set  $U \subset \mathbf{C}^n$ . We want to construct an  $\mathcal{O}_S$ -algebra homomorphism  $\phi : \mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S; B(K))$ .

(a) Consider first  $S = U' \subset \mathbf{C}^m$ ,  $U'$ -open. If  $h \in \mathcal{O}_{U' \times U}(U' \times U)$  and  $s \in U'$ ,  $x \in K$ , define  $(\phi(h)(s))(x) = h(s, x)$ . Using the Cauchy integral, one can show that  $\phi(h)$  is analytic. On the other hand it's obvious that  $\phi$  is an  $\mathcal{O}_{U'}$ -algebra homomorphism.

(b) Let  $S$  have a special model in the polydisc  $\Delta$  in  $\mathbf{C}^m$ , defined by a sheaf  $\mathcal{I}$  of ideals of  $\mathcal{O}_\Delta$ , and let  $\mathcal{J}$  be generated by  $f_1, \dots, f_p$ ,  $V$ -a polycylinder neighbourhood of  $K$  in  $U$ . By Cartan's theorem  $B$  for a polycylinder,

the sequence  $0 \rightarrow \mathcal{I}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \rightarrow 0$  is exact. If we denote by  $\tilde{\pi}$  the projection  $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K))$ ,  $(f_1, \dots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset \subset \text{Ker } \tilde{\pi}$ . Therefore, because  $\pi$  is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$ , such that the diagram