

## 2. The vanishing theorem of Kodaira

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$h_i \circ h_j^{-1}$  can be written  $(x, \gamma'(x, c))$  where  $\gamma'(x, c) \in \mathbf{C}$ . According to the last property in (iii), for fixed  $x \in U_i \cap U_j$  the mapping  $c \rightarrow \gamma'(x, c)$  is a  $\mathbf{C}$ -isomorphism of  $\mathbf{C}$  onto itself. Therefore

$$\gamma'(x, c) = g_{ij}(x) \cdot c, \text{ where } g_{ij}(x) \neq 0, \quad (1.1)$$

and it is easily seen that  $g_{ij}$  is holomorphic in  $U_i \cap U_j$ .

The functions  $g_{ij}$  obviously satisfy the cocycle conditions

$$g_{ij} g_{jk} g_{ki} = 1 \quad \text{on} \quad U_i \cap U_j \cap U_k, \quad (1.2)$$

$$g_{ij} g_{ji} = 1 \quad \text{on} \quad U_i \cap U_j. \quad (1.3)$$

The  $g_{ij}$  are called transition functions corresponding to the line bundle  $F$ .

Conversely, it is easy to prove (cf. [4], p. 135) that given an open covering  $\{U_i\}$  and functions  $g_{ij}$  without zeros in  $U_i \cap U_j$  which satisfy the cocycle conditions, we can construct a line bundle which has  $g_{ij}$  as transition functions.

Now, let  $F$  be a line bundle over a complex manifold  $X$ , and let  $\pi$  be the corresponding projection. We denote  $\pi^{-1}(a)$  by  $F_a$ . Let  $F_a^*$  be the  $\mathbf{C}$ -dual of  $F_a$ . Then

$$F^* = \bigcup_{a \in X} F_a^*$$

is in a natural way a holomorphic line bundle over  $X$ , which is called the dual bundle of  $F$ . If  $F$  has transition functions  $\{g_{ij}\}$ , then  $F^*$  has transition functions  $\{g_{ij}^{-1}\}$ .

*Definition 1.6.* Let  $F$  be a holomorphic line bundle over a compact complex manifold. Then  $F$  is *negative* if the zero cross section  $\mathfrak{o}$  of  $F$  can be blown down to a point.  $F$  is *positive* if the dual bundle is negative.

In the sequel we let  $\underline{F}$  denote the sheaf of germs of analytic sections of a line bundle  $F$ .

## 2. THE VANISHING THEOREM OF KODAIRA

This is the following theorem, which is our first main result:

*Theorem 2.1.* Let  $X$  be a compact connected complex manifold and  $F$  a positive line bundle on  $X$  and  $S$  a coherent analytic sheaf on  $X$ . Then there exists an integer  $k(S, F)$  such that for  $k > k(S, F)$  we have  $H^q(X, S \otimes \underline{F}^k) = 0$  ( $\forall q \geq 1$ ).

The proof uses the following finiteness theorem:

*Theorem 2.2.* Let  $V$  be a complex manifold,  $S$  a coherent analytic sheaf on  $V$ , and  $D \subset\subset V$  a strictly pseudoconvex subdomain of  $V$ . Then the cohomology groups  $H^q(D, S)$  are finite-dimensional  $\mathbf{C}$ -vector spaces if  $q \geq 1$ .

For a proof of Theorem 2.2 see Section 4.4 of the lectures by Malgrange in these notes.

*Proof of Theorem 2.1.*

Let  $E$  be the dual bundle of  $F$ . By hypothesis,  $E$  is negative. Thus, by Lemma 1.4, the zero cross section of  $E$  has a strictly pseudoconvex neighbourhood  $D$ .

By definition, we have a projection  $\pi: E \rightarrow X$ . We will now use  $\pi$  to “lift”  $S$  to a coherent analytic sheaf  $\tilde{S}$  on  $E$ . To do this, we first consider the sheaf of abelian groups  $\pi^{-1}(S)$  which to any point  $a$  of  $E$  assigns the stalk  $S_{\pi(a)}$ . Since  $S_{\pi(a)}$  and the ring  $\mathcal{O}_a(E)$  of germs of analytic functions at  $a$  both are modules over the ring  $\mathcal{O}_{\pi(a)}(X)$ , we can form the tensor product  $\tilde{S}_a = S_a \otimes \mathcal{O}_a(E)$  over  $\mathcal{O}_{\pi(a)}(X)$ . Then  $\tilde{S}_a$  is a module over  $\mathcal{O}_a(E)$ , and this defines  $\tilde{S}$ . Since  $S$  is coherent,  $\tilde{S}$  is also coherent (cf [3], p. 401).

From Theorem 2.2 it now follows that  $H^q(D, \tilde{S})$  are finite-dimensional  $\mathbf{C}$ -vector spaces for  $q \geq 1$ . We complete the proof of Theorem 2.1 by constructing for every  $N$  a natural injection

$$\sum_{k=0}^N H^q(X, S \otimes \underline{F^k}) \rightarrow H^q(D, \tilde{S}),$$

where the sum is the direct sum as vector spaces. In fact, since  $\dim \sum_{k=0}^N H^q$

$= \sum_{k=0}^N \dim H^q$ , the existence of such injections would imply the existence of the desired integer  $k(S, F)$ .

Let  $a$  be a point of the zero cross section  $\mathfrak{o}$  in the negative bundle  $E$ , and let  $U$  be a neighbourhood of  $a$  such that  $E_U \approx U \times \mathbf{C}$ . Identifying  $a \in \mathfrak{o} \subset E$  with the point  $\pi(a) \in X$ , we denote by  $\mathcal{O}_a(E)$  and  $\mathcal{O}_a(X)$  the rings of germs of analytic functions on  $E$  at  $a$  and on  $X$  at  $a$ , respectively.

To a germ  $f \in \mathcal{O}_a(E)$  corresponds a Taylor series  $\sum_{\nu=0}^{\infty} f_{\nu}(x) z^{\nu}$ , converging in some neighbourhood  $U' \times D_r$ , where  $U' \subset U$  and  $D_r = \{z; |z| < r\}$ .

For  $x \in U$ , let  $e'(x) \in E_x$  correspond to  $(x, 1)$  under the isomorphism  $E_x \approx U \times \mathbf{C}$ , and let  $e(x) \in F_x$  be defined by  $\langle e(x), e'(x) \rangle = 1$ . Then

$e(x)$  is a holomorphic section of  $F$  over  $U$ , and every germ  $p \in \underline{F}_a^k$  is represented by  $p(x) e(x) \otimes e(x) \otimes \dots \otimes e(x)$ , ( $k$  factors  $e(x)$ ), where  $p(x)$  is holomorphic in a neighbourhood of  $a$ . But  $p(x) e(x) \otimes e(x) \otimes \dots \otimes e(x) \in \underline{F}_x^k$  can be identified with the multilinear functional

$$(z_1, \dots, z_k) \rightarrow p(x) z_1 \cdot \dots \cdot z_k$$

and therefore also with the polynomial  $p(x) z^k$ .

Hence, for every  $N$  we obtain an injection

$$i_N: \sum_{k=0}^N \underline{F}_a^k \rightarrow \mathcal{O}_a(E)$$

by mapping  $(p_0, p_1, \dots, p_N) \in \sum_0^N \underline{F}_a^k$  onto the germ at  $a$  of  $\sum_{k=0}^N f_k(x) z^k$ , where  $f_k(x)$  is holomorphic in a neighbourhood of  $a$  and  $f_k(x) z^k$  corresponds to  $p_k \in \underline{F}_a^k$  in the way described above. Further the map  $q_N: \sum_0^\infty f_v(x) z^v \rightarrow \sum_0^\infty f_k(x) z^k$  gives rise to a homomorphism  $\mathcal{O}_a(E) \rightarrow \underline{F}_a^k$  such that  $q_N \circ i_N = \text{id}$ . It is obvious that this mapping  $i_N$  is injective.

From  $i_N$  we also obtain a homomorphism

$$j_N: S \otimes_{\mathcal{O}(X)} \sum_0^N \underline{F}^k \rightarrow S \otimes_{\mathcal{O}(X)} \mathcal{O}(E) = \tilde{S},$$

and the corresponding homomorphism

$$j_N^*: H^q(X, S \otimes \sum_0^N \underline{F}^k) \rightarrow H^q(\mathfrak{v}, \tilde{S}).$$

Further, the map  $q_N$  defined above gives rise to a homomorphism

$$\tilde{S} \rightarrow S \otimes_{\mathcal{O}(X)} \sum_0^N \underline{F}^k,$$

and hence a map

$$\eta_N: H^q(\mathcal{O}, \tilde{S}) \rightarrow H^q(X, S \otimes \sum_0^N \underline{F}^k)$$

such that  $\eta_N \circ j_N^* = \text{id}$ . Hence  $j_N^*$  is injective.

This mapping can be factored as follows

$$H^q(\mathcal{O}, S \otimes \sum_0^N \underline{F}^k) = \sum_0^N H^q(S \otimes \underline{F}^k) \xrightarrow{\alpha} H^q(D, \tilde{S}) \xrightarrow{\beta} H^q(\mathfrak{v}, \tilde{S}),$$

and as  $\beta \circ \alpha$  is an injection,  $\alpha$  also is an injection, which proves the theorem.