2. The vanishing theorem of Kodaira

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$h_i \circ h_j^{-1}$ can be written $(x, \gamma'(x, c))$ where $\gamma'(x, c) \in \mathbf{C}$. According to the last property in (iii), for fixed $x \in U_i \cap U_j$ the mapping $c \to \gamma'(x, c)$ is a $\mathbf{C}$-isomorphism of $\mathbf{C}$ onto itself. Therefore

$$\gamma'(x, c) = g_{ij}(x) \cdot c \,, \text{ where } g_{ij}(x) \neq 0 \,, \qquad (1.1)$$

and it is easily seen that g_{ij} is holomorphic in $U_i \cap U_j$.

The functions g_{ij} obviously satisfy the cocycle conditions

$$g_{ij}\, g_{jk}\, g_{ki} = 1 \quad \text{on} \quad U_i \cap U_j \cap U_k \,, \qquad (1.2)$$

$$g_{ij}\, g_{ji} = 1 \quad \text{on} \quad U_i \cap U_j \,. \qquad (1.3)$$

The g_{ij} are called transition functions corresponding to the line bundle F.

Conversely, it is easy to prove (cf. [4], p. 135) that given an open covering $\{ U_i \}$ and functions g_{ij} without zeros in $U_i \cap U_j$ which satisfy the cocycle conditions, we can construct a line bundle which has g_{ij} as transition functions.

Now, let F be a line bundle over a complex manifold X, and let π be the corresponding projection. We denote $\pi^{-1}(a)$ by F_a. Let F_a^* be the $\mathbf{C}$-dual of F_a. Then

$$F^* = \bigcup_{a \in X} F_a$$

is in a natural way a holomorphic line bundle over X, which is called the dual bundle of F. If F has transition functions $\{ g_{ij} \}$, then F^* has transition functions $\{ g_{ij}^{-1} \}$.

Definition 1.6. Let F be a holomorphic line bundle over a compact complex manifold. Then F is *negative* if the zero cross section $\mathfrak{o}$ of F can be blown down to a point. F is *positive* if the dual bundle is negative.

In the sequel we let $\underline{F}$ denote the sheaf of germs of analytic sections of a line bundle F.

2. The vanishing theorem of Kodaira

This is the following theorem, which is our first main result:

Theorem 2.1. Let X be a compact connected complex manifold and F a positive line bundle on X and S a coherent analytic sheaf on X. Then there exists an integer $k(S, F)$ such that for $k > k(S, F)$ we have $H^q(X, S \otimes \underline{F^k}) = 0 \; (\forall q \geqslant 1)$.

The proof uses the following finiteness theorem:

Theorem 2.2. Let V be a complex manifold, S a coherent analytic sheaf on V, and $D \subset\subset V$ a strictly pseudoconvex subdomain of V. Then the cohomology groups $H^q(D, S)$ are finite-dimensional **C**-vector spaces if $q \geq 1$.

For a proof of Theorem 2.2 see Section 4.4 of the lectures by Malgrange in these notes.

Proof of Theorem 2.1.

Let E be the dual bundle of F. By hypothesis, E is negative. Thus, by Lemma 1.4, the zero cross section of E has a strictly pseudoconvex neighbourhood D.

By definition, we have a projection π: $E \to X$. We will now use π to " lift " S to a coherent analytic sheaf $\tilde{S}$ on E. To do this, we first consider the sheaf of abelian groups $\pi^{-1}(S)$ which to any point a of E assigns the stalk $S_{\pi(a)}$. Since $S_{\pi(a)}$ and the ring $\mathcal{O}_a(E)$ of germs of analytic functions at a both are modules over the ring $\mathcal{O}_{\pi(a)}(X)$, we can form the tensor product $\tilde{S}_a = S_a \otimes \mathcal{O}_a(E)$ over $\mathcal{O}_{\pi(a)}(X)$. Then $\tilde{S}_a$ is a module over $\mathcal{O}_a(E)$, and this defines $\tilde{S}$. Since S is coherent, $\tilde{S}$ is also coherent (cf [3], p. 401).

From Theorem 2.2 it now follows that $H^q(D, \tilde{S})$ are finite-dimensional **C**-vector spaces for $q \geq 1$. We complete the proof of Theorem 2.1 by constructing for every N a natural injection

$$\sum_{k=0}^{N} H^q(X, S \otimes \underline{F}^k) \to H^q(D, \tilde{S}),$$

where the sum is the direct sum as vector spaces. In fact, since $\dim \sum_{k=0}^{N} H^q = \sum_{k=0}^{N} \dim H^q$, the existence of such injections would imply the existence of the desired integer $k(S, F)$.

Let a be a point of the zero cross section $\mathfrak{o}$ in the negative bundle E, and let U be a neighbourhood of a such that $E_U \approx U \times \mathbf{C}$. Identifying $a \in \mathfrak{o} \subset E$ with the point $\pi(a) \in X$, we denote by $\mathcal{O}_a(E)$ and $\mathcal{O}_a(X)$ the rings of germs of analytic functions on E at a and on X at a, respectively.

To a germ $f \in \mathcal{O}_a(E)$ corresponds a Taylor series $\sum_{\nu=0}^{\infty} f_\nu(x) z^\nu$, converging in some neighbourhood $U' \times D_r$, where $U' \subset U$ and $D_r = \{z; |z| < r\}$.

For $x \in U$, let $e'(x) \in E_x$ correspond to $(x, 1)$ under the isomorphism $E_x \approx U \times \mathbf{C}$, and let $e(x) \in F_x$ be defined by $< e(x), e'(x) > = 1$. Then

$e(x)$ is a holomorphic section of F over U, and every germ $p \in \underline{F}_a^k$ is represented by $p(x)\, e(x) \otimes e(x) \otimes \ldots \otimes e(x)$, ($k$ factors $e(x)$), where $p(x)$ is holomorphic in a neighbourhood of a. But $p(x)\; e(x) \otimes e(x) \otimes \ldots \otimes e(x) \in \underline{F}_x^k$ can be identified with the multilinear functional

$$(z_1, \ldots, z_k) \to p(x)\, z_1 \cdot \ldots \cdot z_k$$

and therefore also with the polynomial $p(x)\, z^k$.

Hence, for every N we obtain an injection

$$i_N\colon \sum_{k=0}^{N} \underline{F}_a^k \to \mathcal{O}_a(E)$$

by mapping $(p_0, p_1, \ldots, p_N) \in \sum_0^N \underline{F}_a^k$ onto the germ at a of $\sum_{k=0}^N f_k(x)\, z^k$, where $f_k(x)$ is holomorphic in a neighbourhood of a and $f_k(x)\, z^k$ corresponds to $p_k \in \underline{F}_a^k$ in the way described above. Further the map $q_N\colon \sum_0^\infty f_\nu(x) z^\nu \to f_k(x) z^k$ gives rise to a homomorphism $\mathcal{O}_a(E) \to \underline{F}_a^k$ such that $q_N \circ i_N = \mathrm{id}$. It is obvious that this mapping i_N is injective.

From i_N we also obtain a homomorphism

$$j_N : S \otimes_{\mathcal{O}(X)} \sum_0^N \underline{F}^k \to S \otimes_{\mathcal{O}(X)} \mathcal{O}(E) = \tilde{S},$$

and the corresponding homomorphism

$$j_N^* : H^q(X, S \otimes \sum_0^N \underline{F}^k) \to H^q(\mathfrak{o}, \tilde{S}).$$

Further, the map q_N defined above gives rise to a homomorphism

$$\tilde{S} \to S \otimes_{\mathcal{O}(X)} \sum_0^N \underline{F}^k,$$

and hence a map

$$\eta_N\colon H^q(\mathcal{O}, \tilde{S}) \to H^q(X, S \otimes \sum_0^N \underline{F}^k)$$

such that $\eta_N \circ j_N^* = \mathrm{id}$. Hence j_N^* is injective.

This mapping can be factored as follows

$$H^q(\mathcal{O}, S \otimes \sum_0^N \underline{F}^k) = \sum_0^N H^q(S \otimes \underline{F}^k) \xrightarrow{\alpha} H^q(D, \tilde{S}) \xrightarrow{\beta} H^q(\mathfrak{o}, \tilde{S}),$$

and as $\beta \circ \alpha$ is an injection, α also is an injection, which proves the theorem.