4. Line bundle associated to a divisor

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easily seen that the sequence

$$0 \to H \to \mathcal{O}(X) \to \mathbf{C}_a \oplus \mathbf{C}_b \to 0$$

is exact. From this we conclude as above that there exists an integer $s(a, b)$ such that the sequence

$$\Gamma(X, \underline{E}^{s(a,b)}) \to E_a^{\,s(a,b)} \oplus E_b^{\,s(a,b)} \to 0$$

is exact. Therefore there exists a neighbourhood W of (a, b) in $X \times X$ such that if $(a', b') \in W$, then the sections of $\Gamma(X, \underline{E}^{s(a,b)})$ separate a' and b'; that is, if $\sigma_0, \ldots, \sigma_k$ is a basis of $\Gamma(X, \underline{E}^{s(a,b)})$, then $(\sigma_0(a'), \ldots, \sigma_k(a'))$ and $(\sigma_0(b'), \ldots, \sigma_k(b'))$ are different points in $\mathbf{P}^k$. Let l be a positive integer, let $(a', b') \in W$, and let σ be a section of $\Gamma(X, \underline{E}^{s(a,b)})$ such that $\sigma(a') \neq 0$ and $\sigma(b') \neq 0$. Then $\sigma^{l-1} \otimes \sigma_0, \ldots, \sigma^{l-1} \otimes \sigma_k$ are sections of $\Gamma(X, \underline{E}^{ls(a,b)})$ such that $((\sigma^{l-1} \otimes \sigma_0)(a'), \ldots, (\sigma^{l-1} \otimes \sigma_k)(a'))$ and $((\sigma^{l-1} \otimes \sigma_0)(b'), \ldots, (\sigma^{l-1} \otimes \sigma_k)(b'))$ are different points in $\mathbf{P}^k$.

This means that for every positive integer l the sections of $\Gamma(X, E^{ls(a,b)})$ separate all point pairs in W. Thus, covering $X \times X - U$ by finitely many such neighbourhoods and taking s'' to be the product of the corresponding $s(a, b)$, we find that the sections of $\Gamma(X, \underline{E}^{s''})$ separate all point pairs in $X \times X - U$.

Let $\alpha = s's''$ and let $\sigma_0, \ldots, \sigma_d$ be a basis of $\Gamma(X, \underline{E}^{\alpha})$. We claim that the mapping f from X into $\mathbf{P}^d$ defined by $f(x) = (\sigma_0(x), \ldots, \sigma_d(x))$ is a biholomorphic imbedding of X into $\mathbf{P}^d$. That this mapping is regular follows from the fact that α is a multiple of s'. What remains to be proved is that the mapping is injective.

Suppose $a, b \in X$, $a \neq b$. If $(a, b) \in U$, then $a, b \in U_i$ for some i, and since α is a multiple of s', we have $f(a) \neq f(b)$. If $(a, b) \in X \times X - U$, then $f(a) \neq f(b)$ since α is a multiple of s''. This proves the theorem.

4. Line bundle associated to a divisor

Let X be a complex manifold and D an analytic subset of X of pure codimension 1 at every point. Such a set D is called a *divisor* of X. We shall construct a line bundle F on X, associated to D.

To do this, we observe that every point of X has a neighbourhood U in which there is a holomorphic function s such that $U \cap D = \{ x \in U; s(x) = 0 \}$, and s generates, at every point of U, the ideal of germs of holomorphic functions vanishing on D. Thus we get a covering of X by open sets U_j and corresponding holomorphic functions s_j. The functions $g_{ij} = s_i/s_j$ are then holomorphic and $\neq 0$ on $U_i \cap U_j$ and $g_{ij}\, g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. The functions g_{ij} therefore define a line bundle F on X with transition functions g_{ij} (see sect. 1). This bundle F is determined by D uniquely up to isomorphism.

If $f \in \Gamma(X, F)$, then the isomorphism $F \mid U_j \simeq U_j \times \mathbf{C}$ gives a holomorphic function f_j on U_j corresponding to f. The functions f_j are related by $f_i = g_{ij} f_j$ on $U_i \cap U_j$. Conversely, if f_j are holomorphic functions on U_j, satisfying this condition, then there is a section f of F on X, which corresponds to f_j on U_j. In particular, the s_j define a section s_D of F on X, and we have $D = \{ x \in X;\ s_D(x) = 0 \}$.

Example. Let $X = \mathbf{P}^n$, and let H be the hyperplane defined in the homogeneous coordinates $z_0, \ldots, z_n$ by $z_0 = 0$. Then the process above associates to H a line bundle F on $\mathbf{P}^n$. As defining functions we can use $s_j(z_0, \ldots, z_n) = z_0/z_j$ on the set U_j where $z_j \neq 0$, $(j=0, \ldots, n)$. We shall prove that F is positive.

Each homogeneous coordinate z_k defines a section $s^{(k)}$ of F, which on each U_j corresponds to the holomorphic function z_k/z_j, for the transition functions are $g_{ij} = s_i/s_j = z_j/z_i$ and we have $z_k/z_i = (z_k/z_j)\, g_{ij}$. Now any section of F can be regarded as a holomorphic function on $E = F^*$, which is linear on the fibres of E. In particular, $s^{(0)}, \ldots, s^{(k)}$ give a holomorphic mapping $\varphi \colon E \to \mathbf{C}^{n+1}$. It is clear that the zero section in E is equal to $\varphi^{-1}(0)$. It is seen by direct verification that φ maps E onto $\mathbf{C}^{n+1}$ and $E - \varphi^{-1}(0)$ biholomorphically onto $\mathbf{C}^{n+1} - \{ 0 \}$. Hence E is negative and F is positive (see sect. 1).

If V is a submanifold of $\mathbf{P}^n$, then the restriction of F to V is a positive line bundle associated to the hyperplane section $D = V \cap H$. In fact, the dual of the restriction is the restriction $E \mid V$ of E to V, and we can use the restriction of φ to $E \mid V$ as " blowing down mapping ".

Let again X be a complex manifold, D a divisor of X, and F the line bundle on X, associated to D. What are the sections of F^k?

If $U \in \Gamma(X, F^k)$, then s is represented in local coordinates on U_j by a holomorphic function f_j. The f_j are connected by $f_i = g_{ij}^k f_j$ on $U_i \cap U_j$, because the functions g_{ij}^k are transition functions for F^k. Now $s_i^k = g_{ij}^k s_j^k$ on $U_i \cap U_j$, the s_i being local equations for the set D as above, and thus $f_i/s_i^k = f_j/s_j^k$ on $U_i \cap U_j$. Hence there exists a meromorphic function f on X such that $f_j = s_j^k f$ on U_j.

This means that f is meromorphic with poles only on D and of order $\leqslant k$. Conversely, if f is such a meromorphic function, then $f_j = s_j^k f$ are holomorphic on U_j and satisfy $f_i = g_{ij}^k f_j$ on $U_i \cap U_j$. Therefore they give a section s of F^k. This correspondence is obtained simply by associating to the section u of F^k, the meromorphic function $u \otimes s_D^{-k}$.

Let us consider again the space $\mathbf{P}^n$ and the bundle F associated to a hyperplane section. Let $(z_0, \ldots, z_n)$ denote homogeneous coordinates for $\mathbf{P}^n$. If $u \in \Gamma(\mathbf{P}^n, F^k)$, u defines, for $z \in \mathbf{P}^n$, an element of $F_z = (E_z^*)^k$, E being the dual bundle to F, hence a map of E_z into $\mathbf{C}$ which is homogeneous of degree k. Thus, u defines a map $\hat{u}$ of $E \to \mathbf{C}$, homogeneous of degree k on each fibre. If φ denotes the map of E into $\mathbf{C}^{n+1}$ defined above, $\hat{u}: E \to \mathbf{C}$ is holomorphic, and vanishes on $\varphi^{-1}(0)$, and so defines a holomorphic function v on $\mathbf{C}^{n+1}$ which is homogeneous of degree k (v is holomorphic also at 0 since a continuous function holomorphic outside a point in $\mathbf{C}^{n+1}$, $n \geqslant 1$, is holomorphic also at this point). The Taylor expansion of v about 0 shows that v is a homogeneous polynomial of degree k. Thus, *any* $u \in \Gamma(\mathbf{P}^n, F^k)$ *can be identified with a homogeneous polynomial of degree* k *in the homogeneous coordinates* $(z_0, \ldots, z_n)$ [i.e. the sections $s^{(0)}, \ldots, s^{(n)}$ of F defined above].

As an application of the vanishing theorem of Kodaira, we now prove the following result due to Chow (cf. [3], p. 170).

Theorem 4.1. Let A be a subvariety of $\mathbf{P}^n$. Then there exist homogeneous polynomials $f_1, \ldots, f_k$ such that $A = \{a \in \mathbf{P}_n; f_1(a) = \ldots = f_k(a) = 0\}$.

Proof. We first prove that if $b \notin A$, then there exists a homogeneous polynomial f vanishing on A with $f(b) \neq 0$. Let S be the sheaf of germs of holomorphic functions vanishing on A and let I be the sheaf of germs of holomorphic functions vanishing at b. Let F be the line bundle associated to a hyperplane section of A. Then F is positive. We get an exact sequence

$$0 \to I \otimes S \otimes F^m \to S \otimes F^m \to S_b \otimes F_b^m \to 0 .$$

By the vanishing theorem of Kodaira, part of the corresponding cohomology sequence will be

$$H^o(\mathbf{P}^n, S \otimes F^m) \to H^o(\mathbf{P}^n, S_b \otimes F_b^m) \to 0 ,$$

if m is sufficiently large. Thus there exists $f \in H^0(P^n, S \otimes F^m)$ which is not zero at b. Since $S \subset \mathcal{O}$, we may look upon $H^0(S \otimes F^m)$ as a subspace of $H^0(F^m)$. It is then the subspace of those sections of $H^0(F^m)$ which vanish on A. Since $f \in H^0(\mathbf{P}^n, F^m)$, this gives the desired homogeneous polynomial.

To prove the theorem, it now suffices to consider all homogeneous polynomials which vanish on A without being identically zero and apply the Hilbert basis theorem.

5. Meromorphic forms

Let X be a complex manifold. A holomorphic differential form is a form which in local coordinates can be written as a finite sum

$$\omega = \Sigma a_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k} \tag{5.1}$$

with holomorphic coefficients $a_{i_1 \dots i_k}$.

A form is called meromorphic if it has locally the form (5.1) with coefficients that are meromorphic functions. Every meromorphic function can be written locally as $f\omega$ where f is a meromorphic function and ω a holomorphic form. The exterior differentiation d, satisfying $d^2 = 0$, extends naturally to meromorphic forms.

Let D be a divisor of X and let $\Omega^p(k, D) = \Omega^p(X, k, D)$ be the sheaf of germs of meromorphic p-forms on X with poles only on D and of order $\leqslant k$, and let $\Omega^p = \Omega^p(X)$ be the sheaf of germs of holomorphic p-forms on X.

Lemma 5.1. There is a natural isomorphism

$$\Omega^p(k, D) \simeq \Omega^p \otimes \underline{F}^k .$$

Proof. A germ in $\Omega^p(k, D)$ at $a \in X$ is represented by a form $f\omega$, where f is a meromorphic function in a neighbourhood U of a, with poles only on D and of order $\leqslant k$, and ω is a holomorphic form on U. Now to f corresponds biuniquely a section $s \in \Gamma(U, F^k)$ (see Sect. 4), which gives a germ $s_a \in \underline{F}^k_a$. Also ω defines a germ $\omega_a \in \Omega^p_a$.

The desired mapping $\Omega^p(K, D) \to \Omega^p \otimes \underline{F}^k$ is now uniquely defined by

$$f\omega \to \omega_a \otimes s_a .$$

To see that it is an isomorphism, it is sufficient to observe that the inverse mapping of $\Omega^p \otimes \underline{F}^k$ into $\Omega^p(k, D)$ is induced by the bilinear mapping $\Omega^p \oplus \underline{F}^k \to \Omega^p(k, D)$, which is given by

$$(\omega_a, s_a) \to (f\omega)_a , \qquad (a \in X) .$$