REPRESENTATIONS OF COMPACT GROUPS AND SPHERICAL HARMONICS

Autor(en): **Coifman, R. R. / Weiss, Guido**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-42346

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

REPRESENTATIONS OF COMPACT GROUPS AND SPHERICAL HARMONICS

by R. R. COIFMAN and Guido WEISS [1]

To the memory of Jean Karamata [2]

§ 1. INTRODUCTORY REMARKS

Special functions (in particular, spherical functions) associated with compact groups have been introduced by many authors. See, for example, E. Cartan [4], Dieudonné [5], Godement [6], Vilenkin [11], Weyl [12]. The principal motivation of these authors has been to extend classical results. Our purpose, on the other hand, is to show how these classical results can be obtained in simple and elegant ways by making use of the basic tools of the theory of representations of compact groups. In the usual treatments of the properties of special functions that we derive (see, for example, Bateman et al. [1]) much use is made of the theory of functions and other analytical tools. We do not use the theory of functions at all. For that matter, very little else in analysis is used, and, given the few basic facts of the theory of representations of compact groups listed below, our development is of an elementary algebraic nature. We refer the reader to the fourth chapter of Stein and Weiss [10] for a development of some of these classical results that exploits, in a somewhat different way, the action of the rotation group $SO(n)$ on n-dimensional Euclidean space $\mathbf{R}^n$.

This article is of an expository nature. Probably, few of the results obtained are new. Moreover, some of the methods that we use are known. On the other hand, this treatment of spherical harmonics is not readily available. Yet, it is not solely because of this last mentioned fact that we feel this article should be published; three other reasons motivated our efforts. First, the theory developed is especially elegant. Secondly, many seemingly unrelated topics are brought together. For example, two

---

[1] This work was supported by U. S. Army Contract DA-31-124-ARO(D)-58.

[2] Le volume 15 (1969) sera entièrement dédié à la mémoire du Professeur J. Karamata.

different inner products that are often used in the space of spherical harmonics of degree k (to be defined later) are shown to be constant multiples of each other (this is relation (3.17) of § 3). Perhaps, in this sense, we introduce new material. Thirdly, this development can be of use as a guide to those who want to study more abstract problems in the theory of compact groups.

We would like to take this opportunity to thank Mrs. Mei Chen and Mr. Edward Wilson who have read the manuscript and made several useful suggestions.

Aside from the standard theorems in measure theory, we shall assume that the reader is familiar with those facts that are usually associated with the Peter-Weyl theorem. More precisely, we shall state without proof theorems (1.1) and (1.3) below. We refer the reader to Pontriagin [7] or Pukanszky [8] for these proofs.

Suppose G is a compact group. A *representation* of G is a continous map, $u \to T(u)$, of G into the class of unitary or orthogonal [1]) operators (depending on whether we are dealing with the complex or real case) on a Hilbert space H that satisfies the relation $T(uv) = T(u)\,T(v)$ for all u and v in G. We shall sometimes write T_u instead of $T(u)$.

$L^2(G)$ denotes the space of all complex valued functions f on G satisfying

$$\int_G | f(u) |^2 \, du < \infty ,$$

where du is the element of Haar measure on G, which we assume to be so normalized that $\int_G du = 1$. We adopt the usual convention of also letting the symbol $L^2(G)$ denote the Hilbert space of all equivalence classes of square integrable function on G, where two such functions are said to be equivalent provided they are equal almost everywhere. When $H = L^2(G)$ the mapping $u \to R_u$, where $[R_u f](v) = f(u^{-1} v)$ is easily seen to be a representation of G; it is called the *(left) regular representation* of G. The function f_u whose value at v is $f(u^{-1} v)$ is called the (left) *translate of* f by u. Thus, $R_u f = f_u$ for $f \in L^2(G)$ and u in G.

If the representation T acts on the Hilbert space H, a subspace $M \subset H$ is said to be *invariant* under the action of T if $T_u s \in M$ for all $u \in G$ whenever s belongs to M. It follows immediately from the facts that T_u is unitary and that the adjoint, T^*_u, of T_u is $T_{u^{-1}}$, that $M^\perp$, the orthogonal

---

[1]) In the usual definition of the notion of a representation the operators are merely assumed to be bounded and invertible. We have defined what is called a *unitary orthogonal representation* of G. Since we shall consider only such representations, our definition avoids the continuous repetition of the words " unitary " and " orthogonal ".

complement of M, is invariant whenever M is invariant. If $\{0\}$ and H are the only invariant subspaces, then the representation T is said to be *irreducible.* A basic result in the theory of representations of compact groups is

Theorem (1.1). *If the representation* T, *acting on the Hilbert space* H, *is irreducible then* H *is finite dimensional.*

Suppose $\{e_1, e_2, \ldots, e_d\}$ is an ortho-normal basis of the Hilbert space H of dimension d and L a linear transformation of H into itself. The matrix $A = (a_{ij})$ of L with respect to this basis is defined by the equations

$$Le_i = \sum_{j=1}^{d} a_{ji} e_j, \qquad i = 1, 2, \ldots, d;$$

thus, the i^{th} column of A consists of the coefficients needed to express Le_i in terms of the basis $\{e_1, e_2 \ldots, e_d\}$. $A^* = (\overline{a_{ji}})$ denotes the adjoint matrix (the matrix of the adjoint transformation, L^*, defined by the relation $(Ls, t) = (s, L^*t)$ [1]) for all s, t in H).

Thus, if L is unitary $AA^* = I = A^*A$, where I is the identity matrix. $A' = (a_{ji})$ denotes the transpose of $A = (a_{ij})$ (in the real case $A' = A^*$). Finally, $tr\,A$ is the trace of A; that is, $tr\,A = \sum_{j=1}^{d} a_{jj}$.

If T is an irreducible representation acting on H, we can choose an orthonormal basis of H, which must be finite by (1.1), and express T as a unitary matrix (t_{ij}) with respect to this basis. In order to avoid using too much notation we will let the symbol T represent the matrix (t_{ij}) as well. The mapping $u \to T(u) = (t_{ij}(u))$ will then be called a (unitary) matrix valued representation and the fact that multiplication is preserved under this mapping can be expressed by the formula

$$t_{ij}(uv) = \sum_{l=1}^{d} t_{il}(u)\, t_{lj}(v) \tag{1.2}$$

for all $u, v \in G$. More generally, a *matrix valued representation* is a continuous mapping that assigns to each $u \in G$ a unitary $d \times d$ matrix $T(u) = (t_{ij}(u))$ in such a way that (1.2) is satisfied. If $\mathbf{C}$ denotes the complex number system and $\mathbf{C}^d$ denotes the d-dimensional complex Euclidean space $\{z = (z_1, z_2, \ldots, z_d) : z_j \in \mathbf{C}, j = 1, 2, \ldots, d\}$ with the usual inner product $z \cdot w = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_d\overline{w_d}$, we also consider $T(u)$

---

[1] Unless otherwise stated, the symbol (s, t) denotes the inner product of s and t.

as the unitary operator mapping $z \in \mathbf{C}^d$ into $w = (w_1, w_2, \ldots, w_d)$, where $w_j = \sum_{l=1}^{d} t_{jl}\, z_l$ for $j = 1, 2, \ldots, d$ (that is, if we regard z and w as column vectors, w is the matrix product $T(u)\, z$). It then follows from (1.2) that $u \to T(u)$ is a representation of G acting on $H = \mathbf{C}^d$ (In the real case we replace $\mathbf{C}$ by $\mathbf{R}$, the real number system, and $\mathbf{C}^d$ by the real Euclidean space $\mathbf{R}^d$).

Suppose T is a matrix valued representation and H_j is the (finite dimensional) subspace of $L^2(G)$ spanned by the entries of the j^{th} column of T. It is an immediate consequence of (1.2) that H_j is invariant under the action of the left regular representation of G.

Two representations S and T, acting on the Hilbert spaces H and K, are said to be *equivalent* when there exists an invertible linear transformation L mapping H onto K such that $T_u L = LS_u$ for all u in G (equivalently, $L^{-1} T_u L = S_u$ for all u in G). A system $\{T^\alpha\}$, $\alpha \in \mathscr{A}$, of irreducible representations of G is said to be *complete* if, given any irreducible representation T, there exists a unique index α such that T and T^α are equivalent. Theorem (1.1), together with the following one, constitute a formulation of the Peter-Weyl theorem:

THEOREM (1.3). *If* $\{\mathrm{T}^\alpha\} = \{(\mathrm{t}_{ij}^\alpha)\}$, $\alpha \in \mathscr{A}$, *is a complete system of irreducible matrix valued representations of the compact group* G, *then the collection of functions* $\sqrt{\mathrm{d}_\alpha}\, \mathrm{t}_{ij}^\alpha$ *is an orthonormal basis of* $\mathrm{L}^2(\mathrm{G})$, *where* d_α *is the dimension of the space* H^α *on which* T^α *acts.*

If T is a representation of G then the function mapping $u \in G$ into $tr\,\{T(u)\} = \chi(u)$ is called the *character of* T. It is clear that if T_1 and T_2 are equivalent representations then the characters of T_1 and T_2 are equal; that is, the character depends only on the equivalence class determined by a representation of G. It is also clear from the orthogonality relations that the character determines the equivalence class of a representation.

COROLLARY (1.4). *Suppose* $\{\mathrm{T}^\alpha\} = \{(\mathrm{t}_{ij}^\alpha)\}$, $\alpha \in \mathscr{A}$, *is a complete system of irreducible matrix valued representations of the compact group* G, f *belongs to* $\mathrm{L}^2(\mathrm{G})$ *and* χ^α *denotes the character of* T^α, *then the series*

$$\sum_{\alpha \varepsilon \mathscr{A}} d_\alpha \int_G f(u)\, \overline{\chi^\alpha(uv^{-1})}\, du = \sum_{\alpha \varepsilon \mathscr{A}} d_\alpha \int_G f(u)\, \chi^\alpha(vu^{-1})\, du$$

converges to f(v) *in the* L^2 *norm* [1].

---

[1] It follows from elementary Hilbert space theory that only a countable number of the summands can be non-zero and the order in which they are taken does not affect the L^2 convergence of the above series.

Proof. By theorem (1.3), the functions $\sqrt{d_\alpha}\, t^\alpha_{ij}$ form an orthonormal basis of $L^2(G)$. Thus,

$$f = \sum_{\alpha \varepsilon \mathscr{A}} \left(\sum_{i,j=1}^{d_\alpha} c^\alpha_{ij}\, t^\alpha_{ij} \right), \tag{1.5}$$

where $c^\alpha_{ij} = d_\alpha \int_G f(u)\, \overline{t^\alpha_{ij}(u)}\, du$ and the convergence is in the L^2 norm. If C^α is the matrix (c^α_{ij}) and $[T^\alpha(v)]'$ is the transpose of $T^\alpha(v)$, then

$$\sum_{i,j=1}^{d_\alpha} c^\alpha_{ij}\, t^\alpha_{ij}(v) = tr\{\, C^\alpha [T^\alpha(v)]' \} = d_\alpha \int_G f(u)\, tr\{\, \overline{T^\alpha(u)}\, [T^\alpha(v)]' \,\}\, du\,.$$

Since $T^\alpha(v)$ is unitary and its inverse is $T^\alpha(v^{-1})$ we have $[T^\alpha(v)]' = \overline{T^\alpha(v^{-1})}$. Thus,

$$d_\alpha \int_G f(u)\, tr\{\, \overline{T^\alpha(u)}\, [T^\alpha(v)]' \,\}\, du = d_\alpha \int_G f(u)\, tr\{\, \overline{T^\alpha(u)\, T^\alpha(v^{-1})} \,\}\, du$$

$$= d_\alpha \int_G f(u)\, tr\{\, \overline{T^\alpha(uv^{-1})} \,\}\, du = d_\alpha \int_G f(u)\, tr\{\, T^\alpha(vu^{-1})\, du$$

and the corollary is proved.

THEOREM (1.6). *Suppose* $\mathrm{T} = (\mathrm{t}_{ij})$, $1 \leqq \mathrm{i}, \mathrm{j} \leqq \mathrm{d}$, *is an irreducible matrix valued representation of* G *and* $\mathrm{H}_j \subset \mathrm{L}^2(\mathrm{G})$ *is the subspace spanned by the entries* $\mathrm{t}_{1j}, \mathrm{t}_{2j}, \ldots, \mathrm{t}_{dj}$ *of the* j^{th} *column of* T. *Then the restriction,* $\mathrm{R}^{(j)}$, *of the left regular representation of* G *to* H_j *is an irreducible representation of* G. *Moreover,* $\mathrm{R}^{(j)}$ *and* $\mathrm{R}^{(k)}$ *are equivalent for* $1 \leqq \mathrm{j}, \mathrm{k} \leqq \mathrm{d}$ *and each of these representations is equivalent to the representation* $\overline{T}$ *on* H *whose value at* $\mathrm{u} \in \mathrm{G}$ *is* $\overline{T}_u = \mathrm{T}'_{u^{-1}}$.

Proof. We have already observed that (1.2) implied that H_j is invariant under the action of the left regular representation. To show that $R^{(j)}$ is irreducible we consider the standard orthonormal basis $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_d = (0, 0, \ldots, 1)$ of $H = \mathbf{C}^d$ and define a linear transformation, L, on H into H_j by putting $Le_i = \sqrt{d}\, t_{ij}$, $1 \leqq i \leqq d$. From the definition we see that $\overline{T}$ is the matrix valued representation having coefficients that are complex conjugates of the ones ocurring in T. By (1.2) we then have

$$(L\overline{T}_u e_i)(v) = L\left(\sum_{l=1}^{d} \overline{t_{li}(u)}\, e_l \right)(v) = \sqrt{d} \sum_{l=1}^{d} \overline{t_{li}(u)}\, t_{lj}(v)$$

$$= \sqrt{d} \sum_{l=1}^{d} t_{il}(u^{-1})\, t_{lj}(v) = \sqrt{d}\, t_{ij}(u^{-1}v) = (R^{(j)}_u\, Le_i)(v)$$

for all $u, v \in G$ and $i = 1, 2, \dots, d$. Thus, $L\overline{T} = R^{(j)} L$ which shows that each of the representations $R^{(j)}$ are equivalent to $\overline{T}$. The theorem now follows immediately.[1])

§ 2. THE CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF SOME SPECIAL GROUPS

In this section we show how one can obtain irreducible representations of some of the classical compact groups. In many cases we describe several representations that are equivalent to each other. We shall see that often one of the members of this equivalence class of representations has special features that make the study of certain properties particularly easy.

If we are given two finite dimensional representations of a compact group G that act on the Hilbert spaces H and K, we can obtain a third representation of G by constructing the *tensor product of* H *and* K. The classical definition of this concept is the following: We choose orthonormal bases $\{ e_1, e_2, \dots, e_m \}$ and $\{ f_1, f_2, \dots, f_n \}$ of H and K, respectively, and we assign to each of the $m \cdot n$ pairs (e_i, f_j) a " product " $e_i \otimes f_j$, called the *tensor product of the elements* e_i and f_j. We then obtain a new Hilbert space by considering all the linear combinations

$$\sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) ,$$

defining addition and scalar multiplication by letting

$$\sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) + \sum_{i,j=1}^{m,n} b_{ij} (e_i \otimes f_j) = \sum_{i,j=1}^{m,n} (a_{ij} + b_{ij}) (e_i \otimes f_j) ,$$

$$c \sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) = \sum_{i,j=1}^{m,n} c a_{ij} (e_i \otimes f_j) ,$$

and the inner product by letting

$$\left(\sum_{i,j=1}^{m,n} a_{ij} (e_i \otimes f_j) , \sum_{i,j=1}^{m,n} b_{ij} (e_i \otimes f_j) \right) = \sum_{i,j=1}^{m,n} a_{ij} \overline{b_{ij}} .$$

This space is denoted by $H \otimes K$ and is called the *tensor product of* H *and* K. It is clear that $\{ e_i \otimes f_j \}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, is an orthonormal basis

---

[1]) We observe that L is an isometry. We will make use of this fact later in § 3.

of $H \otimes K$. If a $= \sum_{i=1}^{m} a_i e_i \in H$ and $b = \sum_{j=1}^{n} b_j f_j \in K$ the tensor product of the elements a and b is defined to be the element $a \otimes b = \sum_{i,j=1}^{m,n} a_i b_j (e_i \otimes f_j)$ of $H \otimes K$.

$H \otimes K$ can be identified with the linear space $\mathscr{L}(H, K)$ of all linear transformations mapping H into K in the following way: to each element $e_i \otimes f_j$ we assign the transformation mapping e_i onto f_j, and e_k, for $k \neq i$, onto the zero vector of K. We then extend this correspondence linearly to all of $H \otimes K$. If we represent the elements of $\mathscr{L}(H, K)$ as $n \times m$ matrices with respect to the two bases in question, this correspondence assigns the matrix $A = (a_{ji})$ to the element $\sum_{i,j=1}^{m,n} a_{ji} (e_i \otimes f_j)$. If $B = (b_{ji})$ is another such matrix, it is easy to check that the inner product of the elements of $H \otimes K$ corresponding to A and B is $tr\,(AB^*)$. We identify $H \otimes K$ with $\mathscr{L}(H, K)$; moreover, we will not use different notation to distinguish the latter space from the corresponding linear space of $m \times n$ matrices.

If $u \in \mathscr{L}(H, H)$ and $v \in \mathscr{L}(K, K)$, we obtain a linear transformation $u \otimes v$ of $H \otimes K$ into itself by letting

$$(u \otimes v)\, t = vtu' \tag{2.1}$$

for all $t \in H \otimes K$ (we are regarding t as a member of $\mathscr{L}(H, K)$ and u' is the transformation whose matrix with respect to $\{ e_1, e_2, \ldots, e_m \}$ is the transpose of the matrix of u). The transformation $u \otimes v$ is called the *tensor product* of u and v. An equivalent way of defining this tensor product is the following one: Suppose

$$ue_i = \sum_{l=1}^{m} a_{li}\, e_l \text{ and } vf_j = \sum_{k=1}^{n} b_{kj} f_k \text{ then we let}$$

$$(u \otimes v)(e_i \otimes f_j) = (ue_i) \otimes (vf_j) = \sum_{l,k=1}^{m,n} a_{li}\, b_{kj} (e_l \otimes f_k) \tag{2.1'}$$

and extend $u \otimes v$ linearly over all of $H \otimes K$.

In order to see that (2.1) and (2.1′) define the same transformation, it clearly suffices to show that they agree when applied to the basis vectors $t_{ij} = e_i \otimes f_j$. From (2.1) we have $(u \otimes v)\, t_{ij} = vt_{ij}\, u'$. Thus,

$$[(u \otimes v)\, t_{ij}]\, e_r = vt_{ij} \sum_{k=1}^{m} a_{rk}\, e_k = va_{ri} f_j = a_{ri} \sum_{k=1}^{n} b_{kj} f_k\,.$$

On the other hand, from (2.1′) we have

$$[(u\otimes v)\, t_{ij}]\, e_r \;=\; \sum_{l,k=1}^{m,n} a_{li}\, b_{kj}\, t_{lk}\, e_r \;=\; \sum_{k=1}^{m} a_{ri}\, b_{kj}\, f_k\,.$$

Thus, in either case we obtain the same transformation.

We now show that if u and v are unitary so is $u \otimes v$. Since $u \otimes v$ is a linear transformation on a finite dimensional Hilbert space it suffices to prove that it is an isometry. But, if $t \in H \otimes K$ we have

$$||\,(u\otimes v)\, t\,||^2 \;=\; \big((u\otimes v)\, t\,,\; (u\otimes v)\, t\big) \;=\; tr\,\{vtu'\,(vtu')\,{}^*\}$$

$$=\; tr\,\{vt\,(u^*u)'\, t^*\, v^*\} \;=\; tr\,\{vtt^*\, v^*\} \;=\; tr\,\{\, tt^*\} \;=\; (t, t) \;=\; ||\,t\,||^2\,.$$

If $u \to S_u$ is a representation of G acting on H and $u \to T_u$ is a representation of G acting on K, then

$$(S_{uv}\otimes T_{uv})\, t \;=\; T_{uv}\, tS'_{uv} \;=\; T_u\, T_v\, tS'_v\, S'_u \;=\; (S_u\otimes T_u)\,(S_v\otimes T_v)\, t$$

for all $u, v \in G$ and $t \in H \otimes K$.

We can summarize these results in the following way:

THEOREM (2.2). *If* $\mathrm{u} \to \mathrm{S}_u$ *and* $\mathrm{u} \to \mathrm{T}_u$ *are two representations of* G *acting on the Hilbert space* H *and* K *respectively, then the mapping* $\mathrm{u} \to \mathrm{S}_u \otimes \mathrm{T}_u$ *is a representation of* G *acting on the tensor product* H $\otimes$ K.

If $H_1, H_2, \ldots, H_k$ are finite dimensional Hilbert spaces we define their *tensor product* $\bigotimes\limits_{j=1}^{k} H_j$ inductively by letting

$$\bigotimes_{j=1}^{k} H_j \;=\; \Big(\bigotimes_{j=1}^{k-1} H_j\Big) \otimes H_k$$

for $k > 2$. We ahsll often write $H_1 \otimes H_2 \otimes \ldots \otimes H_k$ instead of $\bigotimes\limits_{j=1}^{k} H_j$. By making obvious identifications we may regard this product to be associative; the same remark applies to the k-fold tensor products $a_1 \otimes a_2 \otimes \ldots \otimes a_k$, where $a_j \in H_j$ for $1 \leqq j \leqq k$. We shall be interested mostly in the case $H_1 = H_2 = \ldots = H_k = H$ and we shall denote the tensor product of k copies of H by $\mathscr{T}^{(k)}(H)$ or, if there is no chance of confusion, simply by $\mathscr{T}^{(k)}$. We shall fix k for the remainder of this discussion.

If $\{\, e_1\, e_2, \ldots, e_n\}$ is an orthonormal basis of H and Δ is the set of all k-tuples of integers, $m = (m_1, m_2, \ldots, m_k)$, with $1 \leqq m_1, m_2, \ldots, m_k \leqq n$, then the collection $\{ \varepsilon_m \}_{m \,\varepsilon\, \Delta}$, where $\varepsilon_m = e_{m_1} \otimes \ldots \otimes e_{m_k}$, is an orthonormal basis of $\mathscr{T}^{(k)}$. Thus, the general element t of this tensor product has the representation

$$t = \sum_{m \varepsilon \Delta} t_m \, \varepsilon_m ,$$

where the t_m's are complex numbers.

The tensor product $u_1 \otimes u_2 \otimes \ldots \otimes u_k$ of k linear transformations $u_1, u_2, \ldots, u_k$ mapping H into itself can also be defined inductively by extending (2.1′). Its action on the basis elements ε_m is given by

$$(u_1 \otimes u_2 \otimes \ldots \otimes u_k) \, \varepsilon_m = (u_1 \, e_{m_1}) \otimes (u_2 \, e_{m_2}) \otimes \ldots \otimes (u_k \, e_{m_k}) \, .$$

When $u_1 = u_2 = \ldots = u_k = u$ we denote this tensor product by T_u. If

$$\begin{pmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{pmatrix}$$

is the matrix of u with respect to the basis $\{ e_1, e_2, \ldots, e_n \}$ we then have

$$T_u \, \varepsilon_m = \sum_{j \varepsilon \Delta} (T_u)_{j,m} \cdot \varepsilon_j \, , \tag{2.3}$$

where

$$(T_u)_{j,m} = u_{j_1 m_1} u_{j_2 m_2} \ldots u_{j_k m_k}$$

for $j = (j_1, j_2, \ldots, j_k)$ and $m = (m_1, m_2, \ldots m_k)$ in Δ. It follows from theorem (2.2) that the mapping $u \to T_u$ is a representation of the unitary group of transformations on H. This representation acts on $\mathscr{T}^{(k)}$. When $k > 1$ this is not an irreducible representation. In order to exhibit a proper invariant subspace of $\mathscr{T}^{(k)}$ we introduce the subspace $\mathscr{S}^{(k)}$ of *symmetric tensors of degree* k: If τ is a permutation of $\{ 1, 2, \ldots, k \}$ and $m \in \Delta$ we let $\tau m = \{ m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(k)} \}$. Then

$$\mathscr{S}^{(k)} = \{ t = \sum_{m \varepsilon \Delta} t_m \, \varepsilon_m \quad \text{in} \quad \mathscr{T}^{(k)} : t_{\tau m} = t_m$$

for all permutations τ and $m \in \Delta$ $\}$.

THEOREM (2.4). *The subspace* $\mathscr{S}^{(k)}$ *is invariant under the action of the representation* u $\to$ $\mathrm{T}_u = \mathrm{u} \otimes \mathrm{u} \otimes \ldots \otimes \mathrm{u}$ *of the unitary group of transformations on* H.

Proof. We first observe that for any permutation τ of $\{1, 2, \ldots, k\}$ we have

$$(2.5) \qquad (T_u)_{\tau j, m} = (T_u)_{j, \tau^{-1} m}\,.$$

This equality is an immediate consequence of the definition of the coefficients $(T_u)_{j,m}$ (see (2.3)) when τ is a transposition. The general case is then obtained by writing τ as a product of transpositions.

Consider the set of all n tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of non-negative integers satisfying $\|\alpha\| = \alpha_1 + \alpha_2 + \ldots + \alpha_n = k$ and let Δ_α be the set of all the $m = (m_1, m_2, \ldots, m_k)$ in Δ such that i, $1 \leqq i \leqq n$, is one of the components of m precisely α_i times. We then have $\Delta = \bigcup\limits_{\|\alpha\|=k} \Delta_\alpha$ and if $m \in \Delta_\alpha$ then τm also belongs to Δ_α. Moreover, it is easy to see that the collection of all, $\sigma_\alpha = \sum\limits_{m \varepsilon \Delta_\alpha} \varepsilon_m$, $\|\alpha\| = k$, is a basis for $\mathscr{S}^{(k)}$. Consequently, it suffices to show that $T_u \sigma_\alpha \in \mathscr{S}^{(k)}$ when $\|\alpha\| = k$. By (2.3) we have

$$T_u \sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} T_u \varepsilon_m = \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j \varepsilon \Delta} (T_u)_{j,m} \varepsilon_j \Big) = \sum_{j \varepsilon \Delta} \Big(\sum_{m \varepsilon \Delta_\alpha} (T_u)_{j,m} \Big) \varepsilon_j\,.$$

If τ is any permutation, it follows from (2.5) that

$$\sum_{m \varepsilon \Delta_\alpha} (T_u)_{\tau j, m} = \sum_{m \varepsilon \Delta_\alpha} (T_u)_{j, \tau^{-1} m} = \sum_{m \varepsilon \Delta_\alpha} (T_u)_{j,m}\,.$$

Thus, the coefficient of ε_j equals that of $\varepsilon_{\tau j}$ in the above expansion of $T_u \sigma_\alpha$. Hence, $T_u \sigma_\alpha \in \mathscr{S}^{(k)}$ and the theorem is proved.

We shall show that the restriction of this representation $u \to T_u$ to $\mathscr{S}^{(k)}$ is irreducible. This is particularly simple to do if we examine a representation that is equivalent to it that acts on the vector space $\mathscr{P}^{(k)} = \mathscr{P}^{(k,n)}$ of homogeneous polynomial functions of degree k of the n complex variables $z = (z_1, z_2, \ldots, z_n)$. We use the following notation in our discussion of this space: If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of non-negative integers we put $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$ when $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$, $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!$ and, when $\|\alpha\| = k$, $\binom{k}{\alpha} = k!/\alpha!$ (note that $\binom{k}{\alpha}$ is the number of elements in the set Δ_α we introduced in the last proof). The polynomials

$$p_\alpha(z) = \binom{k}{\alpha} z^\alpha = \frac{k!}{\alpha_1! \alpha_2! \ldots \alpha_n!} z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n},$$

$\|\alpha\| = k$, form a basis of $\mathscr{P}^{(k)}$. We have just observed, however, that the elements $\sigma_\alpha = \sum\limits_{m \varepsilon \Delta_\alpha} \varepsilon_m$, $\|\alpha\| = k$, form a basis of the space $\mathscr{S}^{(k)}$ of symmetric tensors of degree k. We can, therefore, extend the map $\sigma_\alpha \to p_\alpha$ linearly and obtain a one to one linear transformation $\pi = \pi^{(k)}$ of $\mathscr{S}^{(k)}$ onto $\mathscr{P}^{(k)}$. This transformation is an isometry if we introduce an inner product on $\mathscr{P}^{(k)}$ by letting $(\pi s, \pi t) = (s, t)$ for all symmetric tensors s and t in $\mathscr{S}^{(k)}$. Obvious consequences of these definitions are: if $p = \sum_{\|\alpha\|=k} c_\alpha p_\alpha$ and $q = \sum_{\|\alpha\|=k} d_\alpha p_\alpha$ then

$$(2.6) \qquad (p, q) = \sum_{\|\alpha\|=k} c_\alpha \bar{d}_\alpha (\sigma_\alpha, \sigma_\alpha) = \sum_{\|\alpha\|=k} c_\alpha \bar{d}_\alpha \binom{k}{\alpha}.$$

On the other hand, if $p(z) = \sum_{\|\alpha\|=k} a_\alpha z^\alpha$ and $q(z) = \sum_{\|\alpha\|=k} b_\alpha z^\alpha$ then

$$(2.6') \qquad (p, q) = \sum_{\|\alpha\|=k} \frac{a_\alpha \bar{b}_\alpha}{\binom{k}{\alpha}}.$$

$$\text{Let } D = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right), \; D^\alpha = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

and, for $p(z) = \sum_{\|\alpha\|=k} a_\alpha z^\alpha$ in $\mathscr{P}^{(k)}$, put

$$p(D) = \sum_{\|\alpha\|=k} a_\alpha D^\alpha .$$

Then, if $q(z) = \sum_{\|\alpha\|=k} b_\alpha z^\alpha$ we have

$$(2.6'') \qquad (p, q) = \frac{1}{k!} \sum_{\|\alpha\|=k} \alpha! \, a_\alpha \bar{b}_\alpha = \frac{1}{k!} p(D) \bar{q},$$

where

$$\bar{q}(z) = \sum_{\|\alpha\|=k} \bar{b}_\alpha z^\alpha . \, ^{1)}$$

THEOREM (2.7). *For each unitary transformation* u *on* H *let* S_u *be the transformation on* $\mathscr{P}^{(k)}$ *that maps a polynomial function* p *into the polynomial function* q (z) = p (u′ z), *where* u′ *is the transpose of the matrix of* u *with respect to the orthonormal basis* $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$. *Then* $\mathrm{S}: \mathrm{u} \to \mathrm{S}_u$ *is a representation of the unitary group of transformations on* H *that is equivalent to* $\mathrm{T}: \mathrm{u} \to \mathrm{T}_u$. *In fact,*

$$(2.8) \qquad \pi T_u = S_u \pi$$

for all unitary transformations u *on* H.

---

1) Il Calderón [2] and in the previously mentioned Chapter IV of Stein and Weiss [10] the inner product on $\mathscr{P}^{(k)}$ was introduced by formula (2.6″). It appears much more natural in this context when we see its connection with the inner product of $\mathscr{S}^{(k)}$.

Proof. Let $L\colon \mathscr{T}^{(k)} \to \mathscr{P}^{(k)}$ be the linear transformation that maps

$$\varepsilon_m = e_{m_1} \otimes e_{m_2} \otimes \dots \otimes e_{m_k} \quad \text{into} \quad z_{m_1} z_{m_2} \dots z_{m_k}\,.$$

Since Δ_α has $\binom{k}{\alpha}$ elements it follows that

$$L\sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} L\varepsilon_m = \binom{k}{\alpha} z^\alpha = p_\alpha(z) = \pi\sigma_\alpha\,.$$

That is, π is the restriction of L to $\mathscr{S}^{(k)}$.

In order to show (2.8) it suffices to show that $\pi T_u \sigma_\alpha = S_u \pi\sigma_\alpha$ for all unitary transformations u on H and $\|\alpha\| = k$. We have, by (2.3),

$$T_u \sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} T_u \varepsilon_m = \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j \varepsilon \Delta} (T_u)_{j,m}\, \varepsilon_j \Big)$$

where

$$(T_u)_{j,m}\, \varepsilon_j = u_{j_1 m_1} u_{j_2 m_2} \dots u_{j_k m_k}\, \varepsilon_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_k}\,.$$

Thus,

$$LT_u \sigma_\alpha = \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j_1, \dots, j_n = 1}^{n} u_{j_1 m_1} \dots u_{j_k m_k} z_{j_1} \dots z_{j_k} \Big)$$

$$= \sum_{m \varepsilon \Delta_\alpha} \Big(\sum_{j=1}^{n} u_{j m_1} z_j \Big) \Big(\sum_{j=1}^{n} u_{j m_2} z_j \Big) \dots \Big(\sum_{j=1}^{n} u_{j m_k} z_j \Big) = p_\alpha(u' z) = S_u \pi\sigma_\alpha\,.$$

Since $T_u \sigma_\alpha \in \mathscr{S}^{(k)}$ by (2.4), we have $LT_u\sigma_\alpha = \pi T_u \sigma_\alpha$. Hence, $\pi T_u \sigma_\alpha = S_u \pi\sigma_\alpha$ for $\|\alpha\| = k$. This shows that (2.8) is true. The fact that $S_{u_1 u_2} = S_{u_1} S_{u_2}$ for any two unitary transformations u_1 and u_2 is immediate. In order to establish the theorem, therefore, we must show that S_u is unitary. But, if p and q belong to $\mathscr{P}^{(k)}$ there exist (unique) symmetric tensors s and t such that $p = \pi s$ and $q = \pi t$. Then,

$$(S_u p, S_u q) = (S_u \pi s, S_u \pi t) = (\pi T_u s, \pi T_u t) = (T_u s, T_u t)$$

$$= (s, t) = (\pi s, \pi t) = (p, q)\,,$$

which shows that S_u is unitary.

THEOREM (2.9). *The representation* $\mathrm{S}\colon \mathrm{u} \to \mathrm{S}_u$ *is irreducible.*

Proof. We first observe that it suffices to show that any linear transformation A on $\mathscr{P}^{(k)}$ such that $AS_u = S_u A$ for all unitary u must be a constant times the identity. To see that this is the case, suppose S leaves a subspace $V \subset \mathscr{P}^{(k)}$ invariant and P is the projection of $\mathscr{P}^{(k)}$ onto V. Since V is also invariant it follows that $PS_u = S_u P$ for all unitary transformations u on H. Consequently, P must be a constant times the identity transformation on $\mathscr{P}^{(k)}$. But, since P is a projection, this constant must be either 0 or 1; thus, V is either $\{0\}$ or $\mathscr{P}^{(k)}$, which means that S is irreducible.

Suppose, then, that the operator A commutes with the representation S and let u be the unitary operator whose matrix with respect to $\{e_1, e_2, \ldots, e_n\}$ is diagonal with $u_{jj} = e^{i\theta_j}$, $1 \leqq j \leqq n$. Then

$$(S_u p_\alpha)(z) = p_\alpha(u' z) = \tbinom{k}{\alpha} (e^{i\theta_1} z_1)^{\alpha_1} (e^{i\theta_2} z_2)^{\alpha_2} \ldots (e^{i\theta_n} z_n)^{\alpha_n} .$$

If we let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\theta . \alpha = \theta_1 \alpha_1 + \theta_2 \alpha_2 + \ldots + \theta_n \alpha_n$ we can express the action of S_u by the simple formula

$$S_u p_\alpha = e^{i\theta \cdot \alpha} p_\alpha .$$

Suppose A, on the other hand, transforms the basis elements p_α in the following manner

$$Ap_\alpha = \sum_{\| \beta \| = k} a_{\beta\alpha} p_\beta .$$

Since $AS_u = S_u A$ we then must have

$$\sum_{\| \beta \| = k} a_{\beta\alpha} e^{i\theta \cdot \beta} p_\beta = S_u A p_\alpha = AS_u p_\alpha = \sum_{\| \beta \| = k} a_{\beta\alpha} e^{i\theta \cdot \alpha} p_\beta .$$

Consequently, $a_{\beta\alpha} e^{i\theta \cdot \beta} = a_{\beta\alpha} e^{i\theta \cdot \alpha}$ for all n-tuples $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)$. Thus, either, $\alpha = \beta$ or $a_{\beta\alpha} = 0$. It follows that $AP_\alpha = a_{\alpha\alpha} p_\alpha$ for $\| \alpha \| = k$.

If $BA = AB$, where B is a linear transformation satisfying

$$Bp_\alpha = \sum_{\| \beta \| = k} b_{\beta\alpha} P_\beta \quad \text{for} \quad \| a \| = k ,$$

we must have

$$\sum_{\| \beta \| = k} b_{\beta\alpha} a_{\alpha\alpha} p_\beta = BAp_\alpha = ABp_\alpha = \sum_{\| \beta \| = k} b_{\beta\alpha} a_{\beta\beta} p_\beta .$$

Thus, $a_{\alpha\alpha} b_{\beta\alpha} = b_{\beta\alpha} a_{\beta\beta}$. Thus, if we can find such an operator B with $b_{\beta\alpha} \neq 0$ for some α and all β ($\| \beta \| = k$) it would follow that $a_{\alpha\alpha} = a_{\beta\beta}$ for all α and β. This would show that A is a constant times the identity operator and the theorem would be proved. In order to obtain such a B we choose a unitary operator u on H whose matrix with respect to $\{e_1, e_1, \ldots, e_n\}$ has no zero elements in the first column (i.e. u_{j1},

$j = 1, 2, \dots, n$, is not zero). With $\alpha = (k, 0, \dots, 0)$ (that is, $p_\alpha(z) = z_1^k$) we then have by (2.7)

$$(S_u p_\alpha)(z) = p_\alpha(u' z) = (z_1 u_{11} + z_2 u_{21} + \dots + z_n u_{n1})^k$$

$$= \sum_{\|\beta\| = k} u_{11}^{\beta_1} u_{21}^{\beta_2} \dots u_{n1}^{\beta_n} z^\beta .$$

Clearly,

$$b_{\beta\alpha} = u_{11}^{\beta_1} u_{21}^{\beta_2} \dots u_{n1}^{\beta_n} \neq 0$$

for all β satisfying $\|\beta\| = k$ and the theorem is proved.

Since S and T are equivalent representations (theorem (2.7)) we have the following corollary of (2.9):

Corollary (2.10). *The representation* T: $u \to T_u$ *is irreducible.*

When the Hilbert space H is n-dimensional Euclidean space $\mathbf{C}^n$ the group of all unitary operators on H is called the *unitary group on* $\mathbf{C}^n$ and is denoted by $U(n)$. The same notation will be used for the group of matrices of the operators in $U(n)$ with respect to the standard basis $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots e_n = (0, 0, \dots, 1)$. The *special unitary group*, $SU(n)$, is the subgroup of those elements of $U(n)$ having determinant 1. The spaces $\mathscr{S}^{(k)}$ and $\mathscr{P}^{(k)}$ are obviously invariant under the restrictions of the representations T and S to $SU(n)$. It is not hard to show that these restrictions are irreducible representations. By the equivalence (2.8) it suffices to show that this is true for the representation S. But this requires only one simple change in the proof of theorem (2.9): Instead of the equality $a_{\beta\alpha} e^{i\theta\cdot\alpha} = a_{\beta\alpha} e^{i\theta\cdot\beta}$ holding for all n-tuples $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ we obtain this same equality for all n-tuples θ satisfying $\theta_1 + \theta_2 + \dots + \theta_n = 0$ (thus, det $u = 1$). This suffices for obtaining the conclusion that either $\alpha = \beta$ or $a_{\beta\alpha} = 0$. For, if $a_{\beta\alpha} \neq 0$ then $e^{i\theta\cdot\alpha} = e^{i\theta\cdot\beta}$ for all such n-tuples θ. Thus, if r is any real number we must have $e^{ir\theta\cdot(\alpha-\beta)} = 1$ whenever $\theta_1 + \theta_2 + \dots + \theta_n = 0$. But $(\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + \dots + (\alpha_n - \beta_n) = k - k = 0$. This allows us to choose $\theta = \alpha - \beta$ and we obtain

$$e^{ir(\alpha-\beta)\cdot(\alpha-\beta)} = 1$$

for all real numbers r, which can occur only if $\alpha = \beta$. We have shown, therefore, the following corollary:

Corollary (2.11). *The restrictions of* S *and* T *to* SU (n) *are equivalent irreducible representations of* SU (n).

It is clearly not reasonable to expect that the restriction of an irreducible representation of $U(n)$ to a subgroup is also an irreducible representation of the subgroup. If we consider the *orthogonal group* $0(n)$ (i.e. those operators in $U(n)$ whose matrices with respect to $\{e_1, e_2, ..., e_n\}$ have only real entries) and restrict S, or T, to $0(n)$ we do not obtain an irreducible representation. In studying the problem of how the space $\mathscr{P}^{(k)}$ can be decomposed into subspaces that are invariant under the action of S restricted to $0(n)$ it is more natrual to consider the elements of $\mathscr{P}^{(k)}$ to be polynomial functions of n real variables. Thus, if we denote this restriction by S^0 and $x = (x_1, x_2, ..., x_n)$ is a point of n-dimensional real Euclidean space $\mathbf{R}^n$ then $(S^0_u p)(x) = p(u' x)$ for each $u \in 0(n)$ and $p \in \mathscr{P}^{(k)}$. We denote the inner product of two points $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ of $\mathbf{R}^n$ by $x \cdot y = x_1, y_1, + x_1 y_2 + ... + x_n y_n$; $|x| = \sqrt{x \cdot x}$ is then the Euclidean norm of x. Since this inner product is invariant under the action of $0(n)$ (that is, $(ux) \cdot (uy) = x \cdot y$ whenever $u \in 0(n)$) the subspace

$$|x|^2 \mathscr{P}^{(k-2)} = \{p \in \mathscr{P}^{(k)} : p(x) = |x|^2 q(x) \quad \text{with} \quad q \in \mathscr{P}^{(k-2)}\}$$

when $k > 1$, and

$$|x|^2 \mathscr{P}^{(k-2)} = \{0\} \quad \text{when} \quad k = 0{,}1 .$$

is invariant under the action of S^0. Consequently, the orthogonal complement $\mathscr{H}_n^{(k)}$ of this subspace is also invariant. We let $S^{k,n}$ denote the restriction of S^0 to $\mathscr{H}_n^{(k)}$. Thus, for each $u \in 0(n)$, $S_u^{k,n} = S^{k,n}(u)$ is the operator mapping a polynomial $p \in \mathscr{H}_n^{(k)}$ into the polynomial $q = S_u^{k,n} p$ whose value at x is $p(u' x) = p(u^{-1} x)$.

We recall that the differential operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

is called the *Laplacian*. If a function f defined in a region of $\mathbf{R}^n$ satisfies $\Delta f = 0$ then f is called a *harmonic function.*

THEOREM (2.12). *The representation* $\mathrm{S}^{k,n}$ *of* $0(\mathrm{n})$ *is irreducible. The space* $\mathscr{H}_n^{(k)}$ *on which it acts consists of all the harmonic polynomial functions on* $\mathbf{R}^n$ *that are homogeneous of degree* k. $\mathscr{P}^{(k)}$ *is the orthogonal direct sum of* $\mathscr{H}_n^{(k)}$ *and the subspaces*

$$|x|^{2j} \mathscr{H}_n^{(k-2j)} = \{p \in \mathscr{P}^{(k)}: \ p(x) = |x|^{2j} q(x), \ q \in \mathscr{H}^{(k-2j)}\}, \quad 1 \leqq j \leqq k/2 ;$$

moreover, the restriction of S^0 *to each of these subspaces is an irreducible representation of* $0\,(\mathrm{n})$.

Proof. By (2.6″) and the definition of $\mathcal{H}_n^{(k)}$ we must have, for $p \in \mathcal{H}_n^{(k)}$, $0 = k\,!\,(r, p) = r\,(D)\,\bar{p}$ for all $r \in |\,x\,|^2\,\mathcal{P}^{(k-2)}$, $k \geqq 2$ (when $k < 2$ polynomials in $\mathcal{P}^{(k)}$ are obviously harmonic). In particular, if we choose $r\,(x) = |\,x\,|^2\,q\,(x)$ where $q = \Delta p$ we have, since $|x|^2\,q\,(x) = q\,(x)\,|x|^2$, $0 = q\,(D)\,\Delta\bar{p} = (\Delta p, \Delta p)$. But this means that $\Delta p = 0$; thus, p is harmonic. The converse, that each harmonic polynomial that is homogeneous of degree k belongs to $\mathcal{H}_n^{(k)}$, is evident.

If we show that $S^{k,n}$ is irreducible the rest of the theorem follows easily by induction. We shall, in fact, prove the irreducibility of the restriction of $S^{k,n}$ to the *special orthogonal group* $SO\,(n)$ consisting of those orthogonal transformations that have determinant 1 (these transformations are called *rotations* and $SO\,(n)$ is also known as the *rotation group* on $\mathbf{R}^n$). The group $SO\,(n-1)$ can be identified is a natural way with a subgroup of $SO\,(n)$. This can be done by fixing the vector $\mathbf{1} = (0, \ldots, 0, 1)$ in $\mathbf{R}^n$ and considering the subgroup $G \subset SO\,(n)$ of all rotations leaving $\mathbf{1}$ fixed. Each such rotation effects a change in the first $(n-1)$ coordinates of a point of $\mathbf{R}^n$ and can, therefore, be considered a rotation acting on $\mathbf{R}^{n-1}$. We shall write $SO\,(n-1) = G \subset SO\,(n)$.

The theorem will be established if we show that (i) *If* R *is the restriction of the left regular representation of* $\mathrm{SO}\,(\mathrm{n})$ *to a subspace* V *of* $\mathcal{P}^{(k)}$ *then there exists a polynomial* $\mathrm{q} \in \mathrm{V}$ *that is invariant under the action of* $\mathrm{SO}\,(\mathrm{n}-1)$. *That is,* $\mathrm{q}\,(\mathrm{u}^{-1}\,\mathrm{x}) = \mathrm{q}\,(\mathrm{x})$ *for all* $\mathrm{u} \in \mathrm{SO}\,(\mathrm{n}-1)$; (ii) *If* W *is a subspace of* $\mathcal{H}_n^{(k)}$ *consisting of vectors that are invariant under the action of* $\mathrm{SO}\,(\mathrm{n}-1)$ *then the dimension of* W *is* 1.

If $S^{k,n}$ were not irreducible then $\mathcal{H}_n^{(k)}$ would be the direct sum of (at least) two invariant subspaces. By (i) each of these subspaces must contain a vector invariant under $SO\,(n-1)$; but this would contradict (ii).

To show (i) we choose an orthormal basis $\{\,Y_j\,\}$ of V and, for each pair of points $x, y \in \mathbf{R}^n$, we define

$$Z_x\,(y) = \sum_j \overline{Y_j\,(x)}\;Y_j\,(y)\,. \tag{2.13}$$

Then $(p, Z_x) = \sum_j (p,\ Y_j)\ Y_j\,(x) = p\,(x)$ for all $p \in V$. This means that Z_x is the unique element of V representing the linear functional mapping p into $p\,(x)$ and, therefore, Z_x is independent of the orthonormal basis we chose. Since S^0 is a unitary transformation on V the functions whose values at $y \in \mathbf{R}^n$ are $Y_j(u^{-1}y)$ also form an orthonormal basis of V. Thus, by (2.13) and the fact that the definition of $Z_x(y)$ is independent of the basis we chose, $Z_{u^{-1}x}(u^{-1}y) = Z_x(y)$ for all u in $SO(n)$. In particular, Z_1 must be invariant under the action of any u in $SO(n-1)$. This proves (i).

To show (ii) we let p be an invariant polynomial under the action of $SO(n-1)$ and we write

$$p(x_1, x_2, \ldots, x_n) = p(x) = \sum_{j=0}^{k} x_n^{k-j} p_j(\xi),$$

where p_j is homogeneous of degree j in the $n-1$ variables $\xi = (x_1, x_2, \ldots, x_{n-1})$. If $u \in SO(n-1)$ and $u^{-1}x = y = (y_1, y_2, \ldots, y_n)$ then $y_n = x_n$. Thus, by our identification of $SO(n-1)$ with a subgroup of $SO(n)$, $(y_1, y_2, \ldots, y_{n-1}) = \eta = u^{-1}\xi$ and

$$\sum_{j=0}^{k} x_n^{k-j} p_j(\xi) = p(x) = p(u^{-1}x) = \sum_{j=0}^{k} x_n^{k-j} p_j(u^{-1}\xi) = \sum_{j=0}^{k} x_n^{k-j} p_j(\eta)$$

for all real numbers x_n. Consequently, $p_j(\xi) = p_j(u^{-1}\xi)$ for all $\xi \in \mathbf{R}^{(n-1)}$ and $u \in SO(n-1)$. But this clearly means that p_j is a *radial function* (i.e. it depends only on $|\xi| = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$) since, if we are given any two points ξ and η with $|\xi| = |\eta|$, there exists a rotation u such that $\eta = u^{-1}\xi$. On the other hand, p_j being homogeneous of degree j, this means that we must have $p_j(\xi) = c_j |\xi|^j = c_j (x_1^2 + \ldots + x_{n-1}^2)^{j/2}$, where c_j is the value of p_j at any point on the surface of the unit sphere $\Sigma_{n-2} = \{\xi \in \mathbf{R}^{(n-1)}; |\xi| = 1\}$. Since p_j is a polynomial c_j must be zero when j is odd. Thus, after relabeling, we have shown that

$$p(x) = \sum_{0 \leq j \leq k/2} c_j x_n^{k-2j} (x_1^2 + \ldots + x_{n-1}^2)^j .$$

On the other hand, since p is harmonic

$$0 = (\Delta p)(x) = \sum_{1 \leq j \leq k/2} (\alpha_j c_j + \beta_j c_{j-1}) x_n^{k-2j} (x_1^2 + \ldots + x_{n-1}^2)^{j-1} ,$$

where $\alpha_j = 2j(n + 2j - 3)$ and $\beta_j = (k - 2j + 1)(k - 2j + 2)$. Since $\alpha_j \neq 0$ for $1 \leqq j \leqq k/2$ this means that

$$c_j = (-1)^j \frac{\beta_1 \beta_2 \ldots \beta_j}{\alpha_1 \alpha_2 \ldots \alpha_j} c_0$$

for $1 \leqq j \leqq k/2$. This shows, therefore, that $p(x)$ is c_0 times the polynomial

$$(2.14) \qquad x_n^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1 \beta_2 \dots \beta_j}{\alpha_1 \alpha_2 \dots \alpha_j} x_n^{k-2j} (x_1^2 + \dots + x_{n-1}^2)^j$$

and (ii) is proved.

COROLLARY (2.15). *The linear space spanned by the class of all polynomials in* $\mathscr{H}_n^{(k)}$, $\mathrm{k} = 0, 1, 2, \dots$, *restricted to the surface* $\Sigma_{n-1} = \{ \mathrm{x} \in \mathbf{R}^n : |\mathrm{x}| = 1 \}$ *of the unit sphere in* $\mathbf{R}^n$ *is uniformly dense in the space* $\mathrm{C}(\Sigma_{n-1})$ *of continuous functions on* Σ_{n-1}.

Proof. It follows from the Weierstrass approximation theorem that the linear space spanned by the class of all polymonials in $\mathscr{P}^{(k)}$, $k = 0, 1, \dots$, restricted to Σ_{n-1} is uniformly dense in $C(\Sigma_{n-1})$. But it follows from theorem (2.12) that if $p(x)$ is in $\mathscr{P}^{(k)}$ then

$$p(x) = h(x) + |x|^2 q_1(x) + |x|^4 q_2(x) + \dots + |x|^{2l} q_l(x),$$

where $h \in \mathscr{H}_n^k$ and $q_j \in \mathscr{H}_n^{(k-2j)}$, $1 \leqq j \leqq l \leqq k/2$. If $x \in \Sigma_{n-1}$, therefore, $p(x) = h(x) + q_1(x) + q_2(x) + \dots + q_l(x)$. That is, p is a (finite) sum of elements of $\mathscr{H}_n^{(j)}$, $0 \leqq j \leqq k$. The corollary now follows immediately.

The harmonic homogeneous polynomials of degree k (that is, the members of $\mathscr{H}_n^{(k)}$) are called the *solid spherical harmonics of degree k*. Their restrictions to the surface of the unit sphere Σ_{n-1} are called the *spherical harmonics of degree k* (or, sometimes, the *surface spherical harmonics*). If $p(\xi)$ is such a restriction, because of the homogeneity, we obtain the value of the original function at any point $x = |x| \xi$ in $\mathbf{R}^n$ by multiplying $p(\xi)$ by $|x|^k$. In view of this close relationship between the spaces of solid and surface spherical harmonics of degree k we denote both of them by $\mathscr{H}_n^{(k)}$. It will be clear from the context which of the two spaces $\mathscr{H}_n^{(k)}$ is under discussion. Furthermore, we will systematically use the Greek letters $\xi, \eta, \dots$ to denote points of Σ_{n-1}, while $x, y, \dots$ will continue to denote the general points of $\mathbf{R}^n$. The spherical harmonic Z_ξ, $\xi \in \Sigma_{n-1}$, defined by (2.13) (which was shown to be independent of the choice of orthonormal basis of $\mathscr{H}_n^{(k)}$) is called the *zonal harmonic with pole* ξ. It is clear from our discussion that $c_0 = Z_{\mathbf{1}}(\mathbf{1})$ times the expression (2.14) equals $Z_{\mathbf{1}}(x)$.

The following theorem is a basic tool that will be used in the next section in order to show how the spherical harmonics can be obtained from irreducible representations. Before stating it we observe that $\| Z_1 \| = \sqrt{(Z_1, Z_1)} = \sqrt{Z_1(\mathbf{1})} = a_k$. This follows immediately from the fact that Z_1 represents the linear functional mapping $p \in \mathscr{H}_n^{(k)}$ onto $p(\mathbf{1})$; that is, $(p, Z_1) = p(\mathbf{1})$. For, taking $p = Z_1$, we then must have $\| Z_1 \|^2 = (Z_1, Z_1) = Z_1(\mathbf{1})$.

THEOREM (2.16). *Let* $\{ \mathrm{Y}_1, \mathrm{Y}_2, \ldots, \mathrm{Y}_{d_k} \}$ *be an orthonormal basis of* $\mathscr{H}_n^{(k)}$ *the space of (surface) spherical harmonics of degree* k, *such that* $\mathrm{Y}_1 = \mathrm{a}_k^{-1} \mathrm{Z}_1$. *Then, if* $(\mathrm{t}_{ij}(\mathrm{u}))$, $\mathrm{u} \in \mathrm{SO}(\mathrm{n})$, *is the matrix of* $\mathrm{S}_u^{k,n}$ *with respect to this basis, we have*

$$Y_j(u\mathbf{1}) = a_k \overline{t_{j1}(u)} = \sqrt{Z_1(\mathbf{1})}\, \overline{t_{j1}(u)} \tag{2.17}$$

for j = 1, 2, ..., d_k.

Proof. If $p \in \mathscr{H}_n^{(k)}$ is orthogonal to Y_1 we obtain

$$0 = (p, Y_1) = a_k^{-1}(p, Z_1) = a_k^{-1} p(\mathbf{1}).$$

In particular,

$$(i) \qquad Y_i(\mathbf{1}) = 0 \quad \textit{for} \quad i = 2, 3, \ldots, d_k.$$

If $v \in SO(n)$ then the matrix $(t_{ij}(v))$ of $S_v^{k,n}$ is given by

$$(S_v^{k,n} Y_j)(\xi) = Y_j(v^{-1}\xi) = \sum_{i=1}^{d_k} t_{ij}(v)\, Y_i(\xi), \quad 1 \leqq j \leqq d_k.$$

Thus, putting $\xi = \mathbf{1}$ and using (i), we obtain

$$Y_j(v^{-1}\mathbf{1}) = t_{1j}(v)\, Y_1(\mathbf{1}) = a_k \overline{t_{j1}(v^{-1})}, \quad 1 \leqq j \leqq d_k.$$

Letting $u = v^{-1}$ this equality reduces to relation (2.17) and the theorem is proved.

It is not hard to evaluate the constant $a_k^2 = Z_1(\mathbf{1})$. In fact, let

$$q(x) = x_n^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1 \beta_2 \ldots \beta_j}{\alpha_1 \alpha_2 \ldots \alpha_j} x_n^{k-2j} (x_1^2 + x_2^2 + \ldots + x_{n-1}^2)^j$$

be the polynomial (2.14). We showed that $Z_1(\mathbf{1})\, q(x) = Z_1(x)$. Thus,

$$\mathrm{Z}_1(\mathbf{1}) = (\mathrm{Z}_1, \mathrm{Z}_1) = (a_k^2 q, a_k^2 q) = [\mathrm{Z}_1(\mathbf{1})]^2 (q, q).$$

This shows that $a_k^2 = Z_1(\mathbf{1}) = 1 / (q, q)$. On the other hand, the inner product (q, q) is easily evaluated once we observe, after an easy calculation, that the polynomials $x_n^{k-2j}(x_1^2 + x_2^2 + \dots + x_{n-1}^2)^j$ are mutually orthogonal and the square of their norm is $\alpha_1 \alpha_2 \dots \alpha_j / \beta_1 \beta_2 \dots \beta_j$. Hence,

$$a_k^{-2} = 1/Z_1(\mathbf{1}) = 1 + \sum_{1 \leqq j \leqq k/2} \frac{\beta_1 \beta_2 \dots \beta_j}{\alpha_1 \alpha_2 \dots \alpha_j}, \tag{2.18}$$

where $\alpha_j = 2j(n + 2j - 3)$ and $\beta_j = (k - 2j + 1)(k - 2j + 2)$. It can be shown that the last expression equals

$$\prod_{j=0}^{k-1} \left(\frac{2j + n - 2}{j + n - 2}\right);$$

thus, we also have

$$a_k^{-2} = \prod_{j=0}^{k-1} \frac{2j + n - 2}{j + n - 2}.\text{[1]} \tag{2.19}$$

In the next section we shall characterize those irreducible representations of $SO(n)$ that are equivalent to $S^{k,n}$. These will be the representations of class 1 (to be defined in § 3). We shall show that the spaces spanned by the first column of the matrix of these representations with respect to certain orthonormal bases are the same whenever two representations are equivalent. Consequently, we can define $Y_j, j = 1, 2, \dots, d_k$, by formula (2.17) when $(t_{ij}(u))$ is such a matrix and obtain the spaces $\mathscr{H}_n^{(k)}$ directly from the general theory of representations of $SO(n)$.

§ 3. REPRESENTATIONS OF CLASS 1 AND SPHERICAL HARMONICS

In the course of the proof of theorem (2.12) we showed that there was precisely a one dimensional subspace of $\mathscr{H}_n^{(k)}$ whose points invariant under the action of $S^{k,n}$ restricted to $SO(n-1)$. As we shall see, it is this property

---

1) When $n = 4$, for example, $a_k^2 = (k+1)2^{-k}$. For $n = 6$, $a_k^2 = 6(k+2)(k+3)2^{-k}$. The fact that $a_k^2 = Z_1(\mathbf{1}) \leqq 1$ can be shown without any calculation. Equation (2.13) defined a "zonal harmonic" for any subspace of $\mathscr{P}^{(k)}$. If, in this definition, we use the orthonormal basis $\left\{\sqrt{\binom{k}{\alpha}}\, x^{\alpha}\right\}$, $\|\alpha\| = k$, we obtain $(x \cdot y)^k$. The value of this function at $x = \mathbf{1} = y$ is obviously 1. If, on the other hand, we use an orthonormal basis that is a continuation of an orthonormal basis of $\mathscr{H}^{(k)}$ we clearly have $Z_1(\mathbf{1}) \leqq (\mathbf{1} \cdot \mathbf{1})^k = 1$.

that will enable us to identify those irreducible representations of $SO(n)$ that are equivalent to $S^{k,n}$. In order to do this we shall study, more generally, those representations T of a compact group G having the property that there exists a compact subgroup K and a subspace W of the Hilbert space on which T acts such that $T(u)$, for $u \in K$, is the identity transformation when restricted to W. Before doing this, however, we would like to show that there is a close connection between $SO(n)$ and Σ_{n-1}.

To begin with, it is not hard to show that the space $SO(n) / SO(n-1)$ of left cosets $[u] = \{ uw: u \in SO(n), w \in SO(n-1) \}$ can be identified in a natural way with the surface of the unit sphere Σ_{n-1}. The topology on this space is the one induced by the projection $u \to [u]$ of $SO(n)$ into $SO(n) / SO(n-1)$ (i.e. a set in this last space is open if and only if its inverse image is open). Given $u \in SO(n)$ let $x_u = u\mathbf{1}$; if $v \in [u]$ then $v\mathbf{1} = u\mathbf{1}$ since $u^{-1} v \in SO(n-1)$ and, thus, $v\mathbf{1} = (uu^{-1}) v\mathbf{1} = u(u^{-1} v\mathbf{1}) = u\mathbf{1}$. Consequently, the mapping Φ: $[u] \to x_u = u\mathbf{1}$ is well defined. Moreover, it is clear that Φ is continuous, one to one and onto Σ_{n-1}. Since $SO(n) / SO(n-1)$ is compact it follows that Φ is a homeomorphism.

Secondly, the Haar measure of $SO(n)$ can be used in order to obtain the ordinary Lebesgue measure on Σ_{n-1}. This follows from the following result.

THEOREM (3.1). *If* f *is a continuous function on* Σ_{n-1} *and* $\xi_0 \in \Sigma_{n-1}$, *then*

$$\int_{\Sigma_{n-1}} f(\xi)\, d\xi = \int_{SO(n)} f(u\xi_0)\, du ,$$

where dξ *is the element of normalized Lebesgue measure on* Σ_{n-1} *(that is,* $\int_{\Sigma_{n-1}} d\xi = 1$) *and* du *is the element of normalized Haar measure on the group* SO (n).

Proof. The only property of Lebesgue measure on Σ_{n-1} that we need is that it is invariant under the action of rotations. Thus, first using this property, then the fact that Haar measure is normalized and Fubini's theorem we have

$$\int_{\Sigma_{n-1}} f(\xi)\, d\xi = \int_{\Sigma_{n-1}} f(u\xi)\, d\xi = \int_{SO(n)} \{ \int_{\Sigma_{n-1}} f(u\xi)\, d\xi \}\, du$$

$$= \int_{\Sigma_{n-1}} \{ \int_{SO(n)} f(u\xi)\, du \}\, d\xi .$$

Let u_0 be a rotation such that $u_0 \xi_0 = \xi$; then, since the compact group $SO(n)$ must be unimodular, the last integral equals

$$\int_{\Sigma_{n-1}} \{ \int_{SO(n)} f(uu_0 \xi_0)\, du \}\, d\xi = \int_{\Sigma_{n-1}} \{ \int_{SO(n)} f(u\xi_0)\, du \}\, d\xi$$

$$= \{ \int_{SO(n)} f(u\xi_0)\, du \} \{ \int_{\Sigma_{n-1}} d\xi \} = \int_{SO(n)} f(u\xi_0)\, du\,.^{1)}$$

We now turn to the general case described at the beginning of § 3. That is, we suppose G is a compact group and K a closed subgroup. Suppose T is a representation of G acting on a Hilbert space H and $W \subset H$ the subspace of all those vectors in H that are invariant under the action of K; that is,

$$W = \{ s \in H;\ T_u s = s \text{ for } u \in K \}.$$

For example, as was mentioned briefly at the beginning of this section, when $G = SO(n)$, $K = SO(n-1)$, $T = S^{k,n}$ and $H = \mathscr{H}_n^{(k)}$ we showed in the course of the proof of theorem (2.12) that W is the one dimensional subspace generated by the zonal harmonic Z_1.

The restriction of T to K is a representation of this subgroup that acts on H. If we choose an orthonormal basis of H that is an extension of an orthonormal basis of W, then the matrix of $T(u)$ with respect to this basis has the form

$$\begin{pmatrix} I_W & 0 \\ 0 & \tilde{T}(u) \end{pmatrix}$$

for all $u \in K$, where I_W is the matrix of the identity operator on W. The mapping $\tilde{T}: u \to \tilde{T}(u) = \tilde{T}_u$ is a (matrix valued) representatoin of K acting on $W^{\perp}$. Let ν be the normalized Haar measure of the group K. Then, if dimension d of H is greater than 1 it follows from the orthogonality relations of theorem (1.3) that

$$\int_K T(u)\, d\nu(u) = \begin{pmatrix} I_W & 0 \\ 0 & 0 \end{pmatrix}; \tag{3.2}$$

or, equivalently, if $(t_{ij}(u))$ is the matrix of $T(u)$ with respect to the above mantioned basis,

$$\int_K t_{ij}(u)\, d\nu(u) = \begin{cases} \delta_{ij} & \text{if } i,j \leqq c = \dim W\,. \\ 0 & \text{otherwise} \end{cases} \tag{3.2'}$$

---

1) The reader should observe that this proof obviously extends to the case when $SO(n)$ is replaced by anyone of its compact subgroups that act transitively on Σ_{n-1}.

If $v \in G$ it follows from (3.2) and the fact that T is a representation that

(3.3)

$$\int_K T(vu)\, d\nu(u) = T(v) \int_K T(u)\, d\nu(u) = \begin{pmatrix} t_{11}(v) \dots t_{1c}(v)\ 0 \dots 0 \\ t_{21}(v) \dots t_{2c}(v)\ 0 \dots 0 \\ \dots\dots\dots\dots\dots \\ t_{d1}(v) \dots t_{dc}(v)\ 0 \dots 0 \end{pmatrix}.$$

Suppose f is a continuous functions on G that is constant on the left cosets vK; that is, $f(vu) = f(v)$ for all $u \in K$. Then,

$$f(v) = \int_K f(v)\, d\nu(u) = \int_K f(vu)\, d\nu(u).$$

On the other hand, an application of Minkowski's integral inequality and (1.5) gives us

$$\int_K f(vu)\, d\nu(u) = \sum_{\alpha \in \mathscr{A}} \Big(\sum_{i,j=1}^{d_\alpha} c_{ij}^{\alpha} \int_K t_{ij}^{\alpha}(vu)\, d\nu(u)\Big),$$

where $\{ T^\alpha \} = \{ (t_{ij}^\alpha) \}$ is a complete system of irreducible matrix valued representations of G and the convergence is in the norm of $L^2(G)$. Thus, from (3.3) we have

$$f(v) = \sum_{\alpha \in \mathscr{A}} \Big(\sum_{i=1}^{d_\alpha} \sum_{j=1}^{c_\alpha} c_{ij}^{\alpha}\, t_{ij}^{\alpha}(v) \Big) \tag{3.4}$$

where c_α is the dimension of $W_\alpha = \{ s \in H^\alpha \colon T_u^\alpha s = s \text{ for } u \in K \}$ and the convergence is, again, in the norm of $L^2(G)$.

When $K = G$ then the spaces W_α must be zero dimensional. On the other hand, $c_\alpha = d_\alpha$ if K consists of only the identity element of G. When $c_\alpha = 1$ the representation T^α is said to be of *class* 1 *with respect to* K. The following theorem shows that, if $G = SO(n)$ and $K = SO(n-1)$, then an irreducible representation of G is either of class 1 with respect to K or there are no vectors in the space on which the representation acts that are invariant under the action of K.

THEOREM (3.5). *Suppose* T *is an irreducible representation of* SO (n), n $\geqq 3$, *acting on the Hilbert space* H *and* $\mathrm{W} = \{ \mathrm{s} \in \mathrm{H} \colon \mathrm{T}_u \mathrm{s} = \mathrm{s}$ *for* $\mathrm{u} \in \mathrm{SO}(\mathrm{n}-1) \}$. *Then the dimension of* W *is eigher* 0 *or* 1.

Proof. Given $v \in SO(n)$ we claim that there exists $w \in SO(n-1)$ such that $wv\mathbf{1} = v^{-1}\mathbf{1}$. If $v \in SO(n-1)$ we can take w to be the identity.

If $v \notin SO(n-1)$ we first observe that $v\mathbf{1} - v^{-1}\mathbf{1}$ is orthogonal to $\mathbf{1}$ because

$$v\mathbf{1} \cdot \mathbf{1} = \mathbf{1} \cdot v^* \mathbf{1} = \mathbf{1} \cdot v^{-1} \mathbf{1} = v^{-1} \mathbf{1} \cdot \mathbf{1} .$$

Since $n \geqq 3$ we can construct a two dimensional subspace V of $\mathbf{R}^n$ spanned by $v\mathbf{1} - v^{-1}\mathbf{1}$ and another vector orthogonal to $\mathbf{1}$. If we rotate this space about $\mathbf{1}$ by π radians and leave the orthogonal complement of the span of $\mathbf{1}$ and V pointwise fixed, we obtain a transformation $w \in SO(n-1)$ with the desired property.

If we put $u_1 = vwv$ and $u_2 = w^{-1}$ then $u_1, u_2 \in SO(n-1)$ and $v = u_1 v^{-1} u_2$. Thus,

(i) $$T_v = T_{u_1} T_{v^{-1}} T_{u_2} .$$

Suppose $\dim W \geqq 2$. Then we can find two vectors, e_1 and e_2, such that $(e_i, e_j) = \delta_{ij}$ and $T_u e_j = e_j$ for $i, j = 1, 2$ and $u \in SO(n-1)$. Let $t_{ij}(u)$ be the entries of the matrix of T_u with respect to an orthonormal basis of H having e_1 and e_2 as its first and second elements. Then $t_{ij}(u) = \delta_{ij}$ if either i or j is 1 or 2 and $u \in SO(n-1)$. This fact, together with (1.2) imply that $t_{ij}(vu) = t_{ij}(v)$ and $t_{ij}(uv) = t_{ij}(v)$ when $i, j = 1, 2$, $u \in SO(n-1)$ $v \in SO(n)$. From equality (i), therefore, we have $t_{ij}(v) = \overline{t_{ji}(v)}$ for $i, j = 1, 2$ and all $v \in SO(n)$.

In particular,

(ii) $$t_{21}(v) = \overline{t_{12}(v)}$$

and

(iii) $$t_{11}(v) = \overline{t_{11}(v)}$$

for all $v \in SO(n)$.

If H_1 is the space generated by $t_{11}, t_{21}, \ldots, t_{d1}$ it then follows from theorem (1.6) that the span of the left translates of t_{11} is again H_1 (otherwise this span would be a proper invariant subspace of H_1). Thus, there exist a finite number of rotations $u_1, u_2, \ldots, u_m$ and constant $c_1\, c_2, \ldots, c_m$ such that

$$t_{21}(u) = \sum_{j=1}^{m} c_j t_{11}(u_j^{-1} u)$$

for all $u \in SO(n)$. But, by theorem (1.3)

$$\int_{SO(n)} t_{21}(u)\, t_{12}(u)\, du = \int_{SO(n)} \overline{t_{12}(u)}\, t_{12}(u)\, du = 1/d .$$

On the other hand, again using the orthogonality relations of theorem (1.3),

$$\int_{SO(n)} t_{11}(u_j^{-1}u)\,t_{12}(u)\,du = \int_{SO(n)} \overline{t_{11}(u_j^{-1}u)}\,t_{12}(u)\,du =$$

$$= \int_{SO(n)} \sum_{l=1}^{d} t_{1l}(u_j^{-1})\,t_{l1}(u)\,t_{12}(u)\,du =$$

$$= \sum_{l=1}^{d} \overline{t_{1l}(u_j^{-1})} \int_{SO(n)} \overline{t_{l1}(u)}\,t_{12}(u)\,du = 0\,.$$

Hence,

$$\int_{SO(n)} t_{21}(u)\,t_{12}(u)\,du = \sum_{j=1}^{m} c_j \int_{SO(n)} t_{11}(u_j^{-1}u)\,t_{12}(u)\,du = 0\,.$$

We therefore obtain the contradiction $1 / d = 0$ and the theorem is proved.

If $\{T^\alpha\}$, $\alpha \in \mathscr{A}$, is a complete system of irreducible matrix valued representations of $SO(n)$, $n \geqq 3$, some of the T^α's will be of class 1 with respect to $SO(n-1)$. The rest of the T^α's will act on Hilbert spaces having no non-zero vectors invariant under the action of $SO(n-1)$. Let $\mathscr{A}_1 \subset \mathscr{A}$ be the set of α such that T^α is of class 1 with respect to $SO(n-1)$.

We fix such a complete system $\{T^\alpha\}$ of irreducible matrix valued representations of $SO(n)$. Suppose T is a representation equivalent to T^α for some $\alpha \in \mathscr{A}_1$; that is, T is of class 1 with respect to $SO(\mathrm{n}-1)$. If H is the Hilbert space on which T acts then there exists a 1 dimensional subspace that is invariant under the action of $SO(n-1)$. Let $\{Y_1, Y_2, \ldots, Y_d\}$ be an orthonormal basis of H such Y_1 spans this 1 dimensional space and $(t_{ij}(u))$ the matrix of T_u with respect to this basis. Then, by (3.3)

$$\int_{SO(n-1)} T(vu)\,d\nu(u) = \begin{pmatrix} t_{11}(v)\,0 \ldots 0 \\ t_{21}(v)\,0 \ldots 0 \\ \cdots\cdots\cdots\cdots \\ t_{d1}(v)\,0 \ldots 0 \end{pmatrix},$$

whenever $v \in SO(n)$

In particular, if χ is the character of T (see § 1), we have

$$\int_{SO(n-1)} \chi(vu)\,du = t_{11}(v) \tag{3.6}$$

for all $v \in SO(n)$. Since the character χ is the same for all members of the class $[T]$ of representations equivalent to T, (3.6) gives us a definition of t_{11} that does not depend on the representative we choose from $[T]$. The same must therefore be true of the vector space spanned by the left translates of t_{11}. By theorem (1.6) this vector space must be the linear span H_1 of the elements $t_{11}, t_{21}, \ldots, t_{d1}$ of the first column of the matrix (t_{ij}) (for, as was argued in the proof of theorem (3.5), if this were not the case H_1 would have a proper invariant subspace). In the proof of (1.6) we showed, moreover, that the restriction to H_1 of the left regular representation of $SO(n)$ is equivalent to the representation $\overline{T}$ whose matrix with respect to $\{ Y_1, Y_2, \ldots, Y_d \}$ is $\overline{(t_{ij})}$. In fact, we showed that this restriction $R^{(1)}$ equals $L\,\overline{T}L^{-1}$, where L is the isometric linear transformation of H onto H_1 mapping Y_i onto $\sqrt{d}\,t_{i1}$, $1 \leqq i \leqq d$ (see footnote at the end of § 1).

We can apply these arguments to the representation $S^{k,n}$ acting on $\mathcal{H}_n^{(k)}$ since it is of class 1 with respect to $SO(n-1)$; in fact, for Y_1 we can choose $a_k^{-1} Z_1$ where $a_k = \sqrt{Z_1(\mathbf{1})} = \sqrt{(Z_1, Z_1)}$ (see theorem (2.16)). In particular there exists $\alpha = \alpha_k \in \mathcal{A}_1$ such that $S^{k,n}$ is equivalent to T^α. We then obtain the same function t_{11}^α from (3.6) by taking $\chi = \chi^\alpha$ to be either the character of $S^{k,n}$ or of T^α. Thus, $S^{k,n}$ and T^α are (isometrically) equivalent to the restriction of the left regular representation of $SO(n)$ to the vector space generated by the left translates of $\overline{t_{11}^\alpha} = \overline{t^\alpha}$. It follows from (2.17) and (3.6) that

$$Z_1(v\mathbf{1}) / Z_1(\mathbf{1}) = \overline{t^\alpha(v)} = \int_{SO(n-1)} \overline{\chi^\alpha(vu)}\, du$$

for all $v \in SO(n)$. In equation (2.13), which was used in order to define Z_1, we could have chosen an orthonormal basis $\{ Y_1, Y_2, \ldots, Y_{d_k} \}$ of $\mathcal{H}_n^{(k)}$ that consists of real valued functions (recall that the sum on the right is independent of the choice of orthonormal basis); we see, therefore, that Z_1 must be real valued. Consequently, we can omit the bars denoting complex conjugation in the last equality and we obtain

$$(3.7) \qquad Z_1(v\mathbf{1}) / Z_1(\mathbf{1}) = t^\alpha(v) = \int_{SO(n-1)} \chi^\alpha(vu)\, du$$

for all $v \in SO(n)$.

It follows that a function p on Σ_{n-1} is a spherical harmonic belonging to $\mathcal{H}_n^{(k)}$ if and only if

$$(3.8) \qquad p(u\mathbf{1}) = F(u),$$

where F is a finite linear combination of left translates of t^α.

Suppose $p, q \in \mathcal{H}_n^{(k)}$. In § 2 we defined their inner product (p, q)

(see (2.6), (2.6′) and (2.6″)). Since $\mathscr{H}_n^{(k)} \subset L^2(\Sigma_{n-1})$ we can also form the inner product that $L^2(\Sigma_{n-1})$ induces on $\mathscr{H}_n^{(k)}$:

$$(3.9) \qquad < p, q > \; = \int_{\Sigma_{n-1}} p(\xi)\,\overline{q(\xi)}\,d\xi\,.$$

It is not hard to show that these two inner products differ only by a multiplicative constant:

$$(3.10) \qquad < p, q > \; = A_k(p, q)\,,$$

where $A_k = Z_1(\mathbf{1}) / d_k = a_k^2 / d_k$.

In order to show this we choose the orthonormal basis $\{ Y_1, Y_2, \ldots, Y_{d_k} \}$ of theorem (2.16), let $p = \Sigma\, b_j\, Y_j$ and $q = \Sigma\, c_j\, Y_j$. Then,

$$(p, q) = \sum_{j=1}^{d_k} b_j \bar{c}_j\,.$$

But by (2.16), (3.1) and (1.3),

$$< p, q > = \int_{\Sigma_{n-1}} p(\xi)\,\overline{q(\xi)}\,d\xi = \int_{SO(n)} a_k^2 \Big(\sum_{j=1}^{d_k} b_j\, \overline{t_{j1}(u)} \Big) \Big(\sum_{j=1}^{d_k} \bar{c}_j t_{j1}(u) \Big)\, du =$$

$$= a_k^2\, d_k^{-1} \sum_{j=1}^{d_k} b_j \bar{c}_j = a_k^2\, d_k^{-1}\, (p, q)\,.$$

In the discussion following (2.13) we showed that $Z_{v\xi}(v\eta) = Z_\xi(\eta)$ for all $v \in SO(n)$ and $\xi, \eta \in \Sigma_{n-1}$. Thus,

$$< Z_{v\xi}, Z_{v\xi} > \; = \int_{\Sigma_{n-1}} | Z_{v\xi}(\eta) |^2\, d\eta = \int_{\Sigma_{n-1}} | Z_{v\xi}(v\eta) |^2\, d\eta =$$

$$= \int_{\Sigma_{n-1}} | Z_\xi(\eta) |^2\, d\eta = \; < Z_\xi, Z_\xi >\,.$$

Consequently,

$$(3.11) \qquad < Z_\xi, Z_\xi > \; = \; < Z_\eta, Z_\eta >$$

for all $\xi, \eta \in \Sigma_{n-1}$. Using the fact that $p(\eta) = (p, Z_\eta)$ for $p \in \mathscr{H}_n^{(k)}$, (3.10), Schwarz's inequality and (3.11) we obtain

$$| Z_\xi(\eta) | = | (Z_\xi, Z_\eta) | = A_k^{-1} \,|< Z_\xi, Z_\eta >| \leqq$$

$$\leqq A_k^{-1} \sqrt{< Z_\xi, Z_\xi > < Z_\eta, Z_\eta >} =$$

$$= A_k^{-1} < Z_\xi, Z_\xi > \; = A_k^{-1} A_k (Z_\xi, Z_\xi) = (Z_1, Z_1) = Z_1(\mathbf{1})\,.$$

We have shown that

$$(3.12) \qquad | Z_\xi(\eta) | \leqq Z_1(\mathbf{1})$$

for all $\xi, \eta \in \Sigma_{n-1}$.

It is not hard to show that each representation T^α, $\alpha \in \mathscr{A}_1$, is equivalent to one of the representations $S^{k,n}$, for some $k = 0, 1, \ldots$. We assume that $t^\alpha_{11}(v) = t^\alpha(v)$ is the function defined by (3.7); equivalently, we can assume that $(t^\alpha_{ij}(v))$ is the matrix of the representation $T^\alpha(v)$ with respect to an orthonormal basis of H^α whose first element is invariant under $SO(n-1)$. We claim that, under these conditions, the system

$$(3.13) \qquad \bigcup_{\alpha \varepsilon \mathscr{A}_1} \{ \sqrt{d_\alpha}\, t^\alpha_{11}, \ldots \sqrt{d_\alpha}\, t^\alpha_{d_\alpha 1} \}$$

is a complete orthonormal system for the class of functions f in $L^2(SO(n))$ that are constant on the left cosets $v\, SO(n-1)$; that is, $f(vu) = f(v)$ for all $u \in SO(n-1)$. This follows from (3.4). In fact we have

$$(3.14) \qquad f(v) = \sum_{\alpha \varepsilon \mathscr{A}_1} \sum_{i=1}^{d_\alpha} c^\alpha_{i1}\, t^\alpha_{i1}(v)$$

for all such functions f (the convergence is in L^2 and the coefficients c^α_{i1} are those introduced in (1.5)). If we consider those indices $\alpha \in \mathscr{A}_1$ that correspond to some $k = 0, 1, 2, \ldots$ in the manner described above (i.e. $S^{k,n}$, being of class 1, must be equivalent to one of the representations $T^\alpha = T^{\alpha k}$ with $\alpha \in \mathscr{A}_1$) we obtain a subcollection of the orthonormal system (3.13). On the other hand, it follows immediately from (2.15) and (3.1) that this subcollection must consist of the entire complete system (3.13).

We collect these various results in the following theorem:

THEOREM (3.15). *Suppose* $\{ T^\alpha \}$, $\alpha \in \mathscr{A}$, *is a complete system of irreducible representations of* SO (n) *and* $\mathscr{A}_1 \subset \mathscr{A}$ *is the set of all* $\alpha \in \mathscr{A}$ *such that* T^α *is of class* 1 *with respect to* SO (n−1). *We can then find a one to one correspondence* $k \leftrightarrow \alpha_k$ *between the non-negative integers* $k = 0, 1, 2, \ldots$ *and the members* $\alpha\ (=\alpha_k)$ *of* $\mathscr{A}_1$ *such that* $S^{k,n}$ *and* $T^{\alpha k}$ *are equivalent. If* χ_k *is the character of* $S^{k,n}$ *(or* $T^{\alpha k}$*) then the zonal harmonic* Z_1 *of* $\mathscr{H}^{(k)}_n$ *is real valued and satisfies*

$$(3.16) \qquad a_k^{-2} Z_1(v\mathbf{1}) = \int_{SO(n-1)} \chi_k(vu)\, du = t^{(k)}(v)$$

for all $v \in SO(n)$. *A function* p *on* Σ_{n-1} *is a spherical harmonic of degree* k *if and only if* $p(u\mathbf{1}) = F(u)$, $u \in SO(n)$, *where* F *is a finite linear combina-*

tion of left translates of $\mathrm{t}^{(k)}$. *If* (p, q) *is the inner product of* p *and* q *in* $\mathscr{H}_n^{(k)}$ *introduced in* § 2 *and*

$$< p,q > \; = \int_{\Sigma_{n-1}} p(\xi)\, \overline{q(\xi)}\, d\xi$$

is the inner product of p *and* q *regarded as members of* $\mathrm{L}^2(\Sigma_{n-1})$ *then*

$$< p,q > \; = A_k(p,q) \tag{3.17}$$

where $\mathrm{A}_k = \mathrm{Z}_1(\mathbf{1}) / \mathrm{d}_k = \mathrm{a}_k^2 / \mathrm{d}_k$. *If* $\mathrm{p} \in \mathscr{H}_n^{(k)}$ *and* $\mathrm{q} \in \mathscr{H}_n^{(j)}$ *with* $\mathrm{k} \neq \mathrm{j}$ *then* $< \mathrm{p}, \mathrm{q} > \; = 0$. *Suppose* $\{ \mathrm{Y}_1^{(k)}, \mathrm{Y}_2^{(k)}. \ldots, \mathrm{Y}_{d_k}^{(k)} \}$ *is an orthonormal basis of* $\mathscr{H}_n^{(k)}$ *then*

$$\bigcup_{k=0}^{\infty} \{ Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_{d_k}^{(k)} \}$$

is a complete orthonormal system in $\mathrm{L}^2(\Sigma_{n-1})$. *The zonal harmonic* Z_ξ, $\xi \in \Sigma_{n-1}$, *is less than or equal to* $\mathrm{Z}_1(\mathbf{1})$ *in absolute value.*

Perhaps the only fact we did not explicitly prove is that $< p, q > \; = 0$ when $p \in \mathscr{H}_n^{(k)}$ anc $q \in \mathscr{H}_n^{(j)}$ with $k \neq j$. But this is an easy consequence of the Peter-Weyl theorem (1.3) and theorem (3.1) since $\int_{SO(n)} F(u)\, \overline{G(u)}\, du = 0$ when F is a finite linear combination of left translates of $t^{(k)}$ and G a finite linear combination of left translates of $t^{(j)}$ (by (1.2) such left translates are linear combinations of entries in the first column of a matrix of the representation with respect to some orthonormal system whose first element is invariant under $SO(n-1)$). We have not considered the problem of determining the degree of homogeneity k from a given irreducible representation T^α of class 1 with respect to $SO(n-1)$. Perhaps the easiest way of doing this is by observing that the dimension of the space H^α on which T^α acts must be the same as that of $\mathscr{H}_n^{(k)}$ when T^α and $S^{k,n}$ are equivalent. But the dimension d_k of $\mathscr{H}_n^{(k)}$ can be easily calculated in terms of k. From theorem (2.12) and the discussion preceeding it, we see that $\mathscr{P}^{(k)}$ is the direct sum of $\mathscr{H}_n^{(k)}$ and $|x|^2 \mathscr{P}^{(k-2)}$. Since this last space obviously has the same dimension as $\mathscr{P}^{(k-2)}$ it follows that $d_k = \dim \mathscr{P}^{(k)} - \dim \mathscr{P}^{(k-2)}$. By an easy combinatorial argument (see Stein and Weiss [10], Chapter IV, §) we can show that

$$\dim \mathscr{P}^{(k)} = \binom{n+k-1}{k} \tag{3.18}$$

Thus, for $n \geqq 3$,

$$d_k = \dim \mathscr{H}_n^{(k)} = \frac{(k+n-3)!\,(2k+n-2)}{(n-2)!\quad k!}. \tag{3.19}$$

Theorem (3.15) shows us how the spherical harmonics we introduced in § 2 can be obtained from the general theory of representations of compact groups applied to $SO(n)$. We have also obtained several properties of these spherical harmonics by using simple arguments based on this general theory. We claim that essentially all the well-known classical facts concerning these special functions can be obtained by equally simple arguments. In the next section we justify this claim by deriving a number of important results in the theory of spherical harmonics. Our arguments will again be based on the general theory of representations of compact groups.

§ 4. SOME PROPERTIES OF SPHERICAL HARMONICS

The zonal harmonics Z_1 are often expressed in terms of certain polynomial functions $P_n^{(k)}$ restricted to the interval $[-1,1]$ that are called the *ultra spherical* (or Gegenbauer) polynomials. We have already obtained such an expression in § 2. In fact let

$$P^{(k)}(t) = a_k^2 \Big(t^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1 \beta_2 \dots \beta_j}{\alpha_1 \alpha_2 \dots \alpha_j} t^{k-2j} (1-t^2)^j\Big) \tag{4.1}$$

for $-1 \leqq t \leqq 1, \alpha_j = 2j(2j+n-3)$, $\beta_j = (k-2j+1)(k-2j+2)$ and $a_k^2 = Z_1(\mathbf{1})$. If $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \Sigma_{n-1}$ and we put $t = \xi_n$, so that $1-t^2 = \xi_1^2 + \dots + \xi_{n-1}^2$, the expression in parenthesis becomes the polynomial (2.14) exaluated at ξ. The observation we made in the paragraph following the proof of Corollary (2.15) is equivalent to the fact $Z_1(\xi)$ and $P^{(k)}(t)$ are equal. Writing $t = \xi \,.\, \mathbf{1}$ this equality becomes

$$Z_1(\xi) = P^{(k)}(\xi \cdot \mathbf{1}) \,. \tag{4.2}$$

Usually, the ultraspherical polynomials are introduced in one of two ways. One method is to apply the Gram-Scmidt process to the powers $1, t, t^2, \dots$ restricted to the interval $[-1, 1]$ with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(t) \overline{g(t)} (1-t^2)^{\frac{n-3}{2}} dt \,. \tag{4.3}$$

Another definition of the polynomials $P^{(k)}$ involves the k^{th} derivative of $(1-t^2)^{(2k+n-3)/2}$:

$$P^{(k)}(t) = \alpha_{k,n} (1-t^2)^{(3-n)/2} \frac{d^k}{dt^k} (1-t^2)^{(n+2k-3)/2} \tag{4.4}$$

It is not hard to show that the definition (4.1) is equivalent to these two definitions. One way of doing this is by first establishing the following lemma:

LEMMA (4.5). *Suppose* φ *is a continuous function on* $[-1, 1]$ *then*

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot\eta)\, d\xi = c_n \int_{-1}^{1} \varphi(t)(1-t^2)^{\frac{n-3}{2}}\, dt$$

where

$$c_n^{-1} = \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}}\, dt\,.$$

Proof. This lemma is really of a geometrical nature. First, we note that

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot\eta)\, d\xi$$

is independent of η since, if u is a rotation,

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot u\eta)\, d\xi = \int_{\Sigma_{n-1}} \varphi(u^*\xi\cdot\eta)\, d\xi = \int_{\Sigma_{n-1}} \varphi(\xi\cdot\eta)\, d\xi\,.$$

Thus, we can choose $\eta = \mathbf{1}$. Having done this, we can then evaluate the integral of $\varphi(\xi.\mathbf{1})$ over Σ_{n-1} by first integrating over a *parallel perpendicular to* $\mathbf{1}$, $\sigma_\theta = \{\xi \in \Sigma_{n-1} : \xi\,.\,\mathbf{1} = \cos\theta\}$, $0 \leqq \theta \leqq \pi$, and then integrating the function of θ we have obtained over the interval $[0, \pi]$. Since $\varphi(\xi.\mathbf{1}) = \varphi(\cos\theta)$ is constant over this parallel and the Lebesgue measure of σ_θ is $\omega_{n-2}(\sin\theta)^{n-2}$ (where ω_{n-2} is the measure of the surface, Σ_{n-2}, of the unit sphere of $\mathbf{R}^{n-1}$) we must have

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot\mathbf{1})\, d\xi = \tilde{c}_n \int_0^{\pi} \omega_{n-2}\,\varphi(\cos\theta)(\sin\theta)^{n-2}\, d\theta\,.$$

The constant

$$\tilde{c}_n = 1 / \int_{-1}^{1} \omega_{n-2}(\sin\theta)^{n-2}\, d\theta$$

must be introduced since we normalized $d\xi$ so that

$$\int_{\Sigma_{n-1}} d\xi = 1\,.$$

The lemma now follows from the change of variables $t = \cos\theta$.

One of the assertions of theorem (3.15) is that

$$< p, q > \ = \int_{\Sigma_{n-1}} p(\xi)\overline{q(\xi)}\,d\xi = 0 \quad \text{if} \quad p \in \mathscr{H}_n^{(k)} \text{ and } q \in \mathscr{H}_n^{(j)}.$$

If we apply this result to $p(\xi) = Z_1^{(k)}(\xi)$ and $q(\xi) = Z_1^{(j)}(\xi)$, (4.2) and lemma (4.5) then imply

$$\int_{-1}^{1} P^{(k)}(t)\,P^{(j)}(t)\,(1-t^2)^{\frac{n-3}{2}}\,dt = 0 \tag{4.6}$$

when $k \neq j$. Since $P^{(k)}$ is a polynomial of degree k, for $k = 0, 1, 2, \ldots$ we have the following result:

THEOREM (4.7). *The polynomials* $P^{(k)}(t)$, $k = 0, 1, 2, \ldots$, *form a complete orthogonal system in* $L^2(-1, 1)$ *with respect to the inner product* (4.3).

Let

$$Q(t) = (1-t^2)^{(3-n)/2}\,\frac{d^k}{dt^k}\,(1-t^2)^{(n+2k-3)/2}$$

and $R(t)$ a polynomial of degree $\leqq (k-1)$. Then, integrating by parts k times, we obtain

$$\int_{-1}^{1} R(t)\,Q(t)\,(1-t^2)^{\frac{n-3}{2}}\,dt = \int_{-1}^{1} R(t)\,\frac{d^k}{dt^k}\,(1-t^2)^{(n+2k-3)/2}\,dt = 0.$$

In particular, Q is orthogonal to $P^{(j)}$ for $j = 0, 1, \ldots, k-1$. Since $Q(t)$ is of degree k it follows from theorem (4.7) that there must exist a constant $\alpha = \alpha_{k,n}$ such that $P^{(k)}(t) = \alpha_{k,n}\,Q(t)$. This is precisely equality (4.4).

The following result, a useful tool in the theory of singular integrals and partial differential equations (see Calderon and Zygmund [3] and Seeley [9]), is an immediate application of the relation (3.17) between the inner products $< , >$ and $(,)$.

THEOREM (4.8). *If* p *is a harmonic polynomial on* $\mathbf{R}^n$ *that is homogeneous of degree* k *then there exists a constant* $B = B(\|\alpha\|, n)$, *depending only on the dimension* n *and* $\|\alpha\|$, *such that*

$$\int_{\Sigma_{n-1}} |D^\alpha p(\xi)|^2\,d\xi \leqq B(k+1)^{2\|\alpha\|} \int_{\Sigma_{n-1}} |p(\xi)|^2\,d\xi.$$

Proof. Suppose $p(x) = \sum_{||\beta||=k} c_\beta x^\beta$. Since, by assumption, $p \in \mathscr{H}_n^{(k)}$ It follows that any one of its partial derivatives, say $\frac{\partial p}{\partial x_n}$, belongs to $\mathscr{H}_n^{(k-1)}$. If Σ' denotes summation over all $\beta = (\beta_1, \beta_2, ..., \beta_n)$ such that $||\beta|| = k$ and $\beta_n > 0$, then

$$\frac{\partial p}{\partial x_n}(x) = \sum{}' \beta_n c_\beta x^{\beta-1}.$$

Thus, by (2.6''),

$$\left(\frac{\partial p}{\partial x_n}, \frac{\partial p}{\partial x_n}\right)_{(k-1)} = \sum_{||\beta||=k}{}' \frac{\beta!}{(k-1)!} \beta_n |c_\beta|^2.$$

Since $||\beta|| = k$ implies $\beta_n \leqq k$ it follows that

$$(p, p)_{(k)} = \sum_{||\beta||=k} \frac{\beta!}{k!} |c_\beta|^2 \geqq \sum_{||\beta||=k}{}' \frac{\beta!}{(k-1)!} \beta_n \frac{1}{\beta_n k} |c_\beta|^2$$

$$\geqq \frac{1}{k^2} \sum_{||\beta||=k}{}' \frac{\beta!}{(k-1)!} \beta_n |c_\beta|^2 = \frac{1}{k^2}\left(\frac{\partial p}{\partial x_n}, \frac{\partial p}{\partial x_n}\right)_{(k-1)}.$$

Repeating this argument we obtain

$$(D^\alpha p, D^\alpha p)_{(k-||\alpha||)} \leqq \left[\frac{k!}{(k-||\alpha||)!}\right]^2 (p, p)_{(k)}. \tag{4.9}$$

From, (3.17), (4.9), (2.19) [1] and (3.19) we then have

$$\int_{\Sigma_{n-1}} |D^\alpha p(\xi)|^2 d\xi = < D^\alpha p, D^\alpha p > = A_{k-||\alpha||} (D^\alpha p, D^\alpha p)_{(k-||\alpha||)}$$

$$\leqq A_{k-||\alpha||} \left[\frac{k!}{(k-||\alpha||)!}\right]^2 (p, p)_{(k)} =$$

$$= A_{k-||\alpha||} \left[\frac{k!}{(k-||\alpha||)!}\right]^2 A_k^{-1} < p, p > = C(\alpha, n, k) \int_{\Sigma_{n-1}} |p(\xi)|^2 d\xi.$$

Here

$$C(\alpha, n, k) = A_{k-||\alpha||} \left[\frac{k!}{(k-||\alpha||)!}\right]^2 A_k^{-1} =$$

---

[1] We only announced equality (2.19) and have not proved it. The reader can check that its proof is particularly easy when the dimension n is even. Equality (2.18), which was proved, can be used here to obtain essentially the same estimates.

$$= \left(\frac{a_{k-||\alpha||}}{a_k}\right)^2 \left(\frac{d_k}{d_{k-||\alpha||}}\right) \frac{(k!)^2}{[(k-||\alpha||)\,!]^2} =$$

$$= \left\{\prod_{j=k-||\alpha||}^{k-1} \frac{2j+n-2}{j+n-2}\right\} \left\{\frac{(k+n-3)\,!\,(2k+n-2)\,(k-||\alpha||)\,!}{k\,!\,(k-||\alpha||+n-3)\,!\,(2k-2||\alpha||+n-2)}\right\}$$

$$\frac{(k!)^2}{[(k-||\alpha||)\,!]^2}$$

The first product in brackets consists of $||\,\alpha\,|| - 1$ terms, each less than or equal to 2; therefore, it is dominated by $2^{||\alpha||-1}$. The second bracket times the last fraction reduce to

$$\frac{2k+n-2}{2k-2\,||\,\alpha\,||+n-2}\left\{\frac{k\,!\,(k+n-3)\,!}{(k-||\alpha||)\,!\,(k-||\alpha||+n-3)\,!}\right\}.$$

Since $||\,\alpha\,|| \leqq k$ and $3 \leqq n$,

$$\frac{2k+n-2}{2k-2\,||\,\alpha\,||+n-2} \leqq \frac{2\,||\,\alpha\,||+n-2}{n-2} = \frac{2\,||\,\alpha\,||}{n-2} + 1 \leqq 2\,||\,\alpha\,|| + 1\,.$$

The term in brackets, however, consists of the product of $||\,\alpha\,||$ numbers $(k+n-3)\,(k+n-4)\,\ldots\,(k+n-||\alpha||-2)$ times another product, $k\,(k-1)\ldots(k-||\alpha||+1)$, of $||\,\alpha\,||$ numbers. Since each factor is no larger than $k+n-3$, the term in brackets is dominated by

$$(k+n-3)^{2||\alpha||} \leqq (n-2)^{2||\alpha||}\,(k+1)^{2||\alpha||}.$$

Thus,

$$C\,(\alpha, n, k) \leqq 2^{||\alpha||-1}\,(2||\alpha||+1)\,(n-2)^{2||\alpha||}\,(k+1)^{2||\alpha||}$$

and the theorem is proved with $B\,(||\alpha||, n) = 2^{||\alpha||-1}\,(2||\alpha||+1)\,(n-2)^{2||\alpha||}$.

Many classical formulae are easily derived from the general theory we have developed. For example, let us consider the relation

$$(4.10) \qquad P^{(k)}\,(\xi_n)\,P^{(k)}\,(\eta_n) =$$

$$= a_k^2\,c_{n-1} \int_{-1}^{1} P^{(k)}\,(\xi_n\eta_n + (1-\xi_n^2)^{1/2}\,(1-\eta_n^2)^{1/2}\,t)\,(1-t^2)^{\frac{n-4}{2}}\,dt$$

which we shall show to be true for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and

$$\eta = (\eta_1, \eta_2, \ldots, \eta_n) \text{ in } \Sigma_{n-1}$$

(the constant c_{n-1} was introduced in the statement of lemma (4.5)). This is formula (20) on page 177 of the Bateman Manuscript Project [1], Volume 1.

Equality (4.10) can be regarded as a functional equation defining the zonal harmonics Z_1 or the ultraspherical polynomials $P^{(k)}$ (in the same sense that the relation $f(x+y) = f(x) f(y)$ can be regarded as a functional equation defining the exponential functions).

We claim that (4.10), as well as the statement in the last paragraph, are nothing but a transcription of the following theorem:

THEOREM (4.11). *Let* $t^{(k)}$, $k = 0, 1, 2, \ldots$, *be the function defined by* (3.16). *Then,*

$$(i) \qquad t^{(k)}(u_1 v u_2) = t^{(k)}(v)$$

for u_1, u_2 *in* $SO(n-1)$ *and* v *in* $SO(n)$. *Moreover,*

$$(ii) \qquad \int_{SO(n-1)} t^{(k)}(v_1 u v_2)\, du = t^{(k)}(v_1)\, t^{(k)}(v_2)$$

for all $v_1, v_2 \in SO(n)$.

Conversely, suppose t *is a continuous function on* $SO(n)$ *that is not identically zero which satisfies*

$$t(u_1 v u_2) = t(v)$$

for u_1, u_2 *in* $SO(n-1$, v *in* $SO(n)$ *and*

$$\int_{SO(n-1)} t(v_1 u v_2)\, du = t(v_1)\, t(v_2)$$

for all v_1, v_2 *in* $SO(n)$. *Then there exists a non-negative integer* k *such that* $t = t^{(k)}$.

Before proving theorem (4.11) we show that equality (ii) does imply (4.10). In fact, from (2.17) and (4.2)

$$a_k^2\, t^{(k)}(u) = P^{(k)}(u\mathbf{1}.\mathbf{1})$$

for all $u \in SO(n)$ (recall that $t^{(k)}$ and, therefore, Z_1 are real valued. This was shown immediately preceding (3.7)). Thus, (4.11), part (ii), becomes

$$a_k^{-2} \int_{SO(n-1)} P^{(k)}(v_1 u v_2 \mathbf{1}.\mathbf{1})\, du = a_k^{-4} P^{(k)}(v_1 \mathbf{1}.\mathbf{1})\, P^{(k)}(v_2 \mathbf{1}.\mathbf{1})$$

If we put $v_2\,\mathbf{1} = \xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $v_1^*\,\mathbf{1} = v_1'\,\mathbf{1} = \eta = (\eta_1, \eta_2, \ldots, \eta_n)$, then $v_2\,\mathbf{1}\,.\,\mathbf{1} = \xi_n$ and $v_1\,\mathbf{1}\,.\,\mathbf{1} = \mathbf{1}\,.\,v_1^*\mathbf{1} = \eta_n$.

Hence,

$$(4.12) \qquad a_k^2 \int_{SO(n-1)} P^{(k)}(u\xi.\eta)\, du = P^{(k)}(\xi_n) P^{(k)}(\eta_n) .$$

We now write

$$\xi = (1-\xi_n^2)^{1/2}\, \xi' + \xi_n \mathbf{1} \quad \textit{and} \quad \eta = (1-\eta_n^2)^{1/2}\, \eta' + \eta_n \mathbf{1} ,$$

where

$$\xi' = (\xi_1', \xi_2', \ldots, \xi_{n-1}', 0) \quad \textit{and} \quad \eta' = (\eta_1', \eta_2', \ldots, \eta_{n-1}', 0)$$

belong to Σ_{n-1} and are orthogonal to **1** (clearly,

$$\xi_j' = \xi_j / (1-\xi_n^2)^{1/2} \quad \textit{and} \quad \eta_j' = \eta_j / (1-\eta_n^2)^{1/2}$$

when ξ_n and η_n are not ± 1; in which case, $\xi_j' = 0 = \eta_j'$ for $1 \leqq j \leqq n-1$). We shall also denote $(\xi_1', \xi_2', \ldots, \xi_{n-1}')$ and $(\eta_1', \eta_2', \ldots, \eta_{n-1}')$ by ξ' and η'; that is, we identify Σ_{n-2} with those points of Σ_{n-1} having last coordinate 0. Thus, for u in $SO(n-1)$

$$u\,\xi \,.\, \eta = (1-\xi_n^2)^{1/2} (1-\eta_n^2)^{1/2}\, u\,\xi' \,.\, \eta' + \xi_n \eta_n .$$

An application of theorem (3.1) and lemma (4.5), therefore, gives us

$$\int_{SO(n-1)} P^{(k)}(u\xi.\eta)\, du = \int_{\Sigma_{n-2}} P^{(k)}\big((1-\xi_n^2)^{1/2}(1-\eta_n^2)^{1/2}\, \xi' . \eta' + \xi_n \eta_n\big)\, d\,\xi =$$

$$= c_{n-1} \int_{-1}^{1} P^{(k)}\big((1-\xi_n^2)^{1/2}(1-\eta_n^2)^{1/2}\, t + \xi_n \eta_n\big)(1-t^2)^{\frac{n-4}{2}}\, dt .$$

Equality (4.10) now follows from this last one and (4.12).

We now turn to the proof of theorem (4.11). Since $tr\,\{AB\} = tr\,\{BA\}$ for any two matrices A and B, we have $\chi_k(u_1 v u_2 u) = \chi_k(v u_2 u u_1)$. Hence, since the Haar measure of $SO(n-1)$ is both left and right invariant,

$$t^{(k)}(v) = \int_{SO(n-1)} \chi_k(vu)\, du = \int_{SO(n-1)} \chi_k(vu_2uu_1)\, du = \int_{SO(n-1)} \chi_k(u_1vu_2u)\, du =$$

$$= t^{(k)}(u_1 v u_2) .$$

This establishes (i). In order to show (ii) we choose a matrix valued representation equivalent to $S^{k,n}$ in such a way that $t_{11}(v) = t^{(k)}(v)$ for $v \in SO(n)$. We can do this, for example, by choosing an orthonormal basis of $\mathscr{H}_n^{(k)}$ whose first element is $a_k^{-1} Z_1$ (see the discussion preceeding (3.7)). Then, by (1.2),

$$\int_{SO(n-1)} t^{(k)}(v_1 u v_2)\, du = \sum_{l=1}^{d_k} \int_{SO(n-1)} t_{1l}(v_1)\, t_{l1}(uv_2)\, du$$

and, by (3.2),

$$\int_{SO(n-1)} t_{l1}(uv_2)\,du = \begin{cases} t_{11}(v_2) & if\ l = 1 \\ 0 & if\ 1 < l \leqq d_k \end{cases}.$$

We therefore obtain the desired result

$$\int_{SO(n-1)} t^{(k)}(v_1 u v_2)\,du = t_{11}(v_1)\,t_{11}(v_2) = t^{(k)}(v_1)\,t^{(k)}(v_2).$$

We now show the converse. Since $t(vu) = t(v)$ for all $v \in SO(n)$ and $u \in SO(n-1)$ it follows from (3.4) and theorem (3.15) that

$$t(v) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} c_l^{(k)}\, t_{l1}^{(k)}(v),$$

the convergence being in $L^2(SO(n))$ (the $t_{ij}^{(k)}$ *'s* are the entries of the matrix valued representation equivalent to $S^{k,n}$ that we chose when we established equality (ii). On the other hand, the fact that $t(uv) = t(v)$ for all $v \in SO(n)$ and $u \in SO(n-1)$ implies that $c_l^{(k)} = 0$ for $l \neq 1$, since we can apply the same argument that was used in order to establish (3.4) by allowing the first row of $(t_{ij}^{(k)})$ to assume the role that was played by the first column.[1]) Thus,

$$t(v) = \sum_{k=0}^{\infty} c_1^{(k)}\, t_{11}^{(k)}(v) = \sum_{k=0}^{\infty} c_k\, t^{(k)}(v), \tag{4.13}$$

the convergence being in $L^2(SO(n))$. Suppose $c_{k_0} \neq 0$ for some k_0. Then

$$d_k \int_{SO(n)} t^{(k_0)}(v) \Big\{ \int_{SO(n-1)} t(vuw)\,du \Big\}\,dv =$$

$$d_k \int_{SO(n-1)} \Big\{ \int_{SO(n)} t^{(k_0)}(vw^{-1}u^{-1})\,t(v)\,dv \Big\}\,du =$$

$$d_k \int_{SO(n-1)} t^{(k_0)}(vw^{-1})\,t(v)\,dv = d_k \int_{SO(n)} t^{(k_0)}(vuw^{-1})\,t(vu)\,dv =$$

$$d_k \int_{SO(n-1)} \Big\{ \int_{SO(n)} t^{(k_0)}(vuw^{-1})\,t(vu)\,dv \Big\}\,du =$$

$$d_k \int_{SO(n)} \Big\{ \int_{SO(n-1)} t^{(k_0)}(vuw^{-1})\,du \Big\}\,t(v)\,dv =$$

$$d_k \int_{SO(n)} t^{(k_0)}(v)\,t^{(k_0)}(w^{-1})\,t(v)\,dv = d_k^{-1}\,c_{k_0}\,t^{(k_0)}(w^{-1}) = d_k^{-1}\,c_{k_0}\,t^{(k_0)}(w)$$

(recall that $t^{(k)}$ is real valued and, thus, $t^{(k)}(w^{-1}) = \bar{t}^{(k)}(w) = t^{(k)}(w)$).

---

[1]) The reader can verify that this is the case by replacing $T(vu)$ by $T(uv)$ in equality (3.3).

On the other hand,

$$d_k \int_{SO(n)} t^{(k_0)}(v) \left\{ \int_{SO(n-1)} t(vuw)\, du \right\} dv =$$

$$= d_k \int_{SO(n)} t^{(k_0)}(v)\, t(v)\, t(w)\, dv = d_k^{-1}\, c_{k_0}\, t(w)\,.$$

Consequently,

$$c_{k_0}\, t(w) = c_{k_0}\, t^{(k_0)}(w)\,. \quad \text{Since} \quad c_{k_0} \neq 0$$

this implies $t = t^{(k_o)}$
and theorem (4.11) is proved.

The fact that relation (4.10) can be regarded as a functional equation defining the zonal harmonics is not its only significance. The general methods we used in establishing it are connected with the operation of convolution in $L^1\,(SO(n))$, the space of integrable functions on $SO\,(n)$. Suppose f, g belong to this space, then their *convolution* $f * g$ is defined by letting

$$(f * g)(v) = \int_{SO(n)} f(u)\, g(vu^{-1})\, du$$

for all $v \in SO\,(n)$.[1]

Let $\{ T^\alpha \}$, $\alpha \in \mathscr{A}$, be a complete system of irreducible matrix valued representations of $SO\,(n)$. For $f \in L^1\,(SO(n))$ we then define its (matrix valued) *Fourier transform* (or its system of *Fourier coefficients*) by putting

$$\hat{f}\,(\alpha) = \int_{SO(n)} f(u)\, T^\alpha(u^{-1})\, du$$

for $\alpha \in \mathscr{A}$. If f is also square integrable this definition is consistent with the Fourier coefficients introduced in the first section. In fact, it can be easily shown that Corollary (1.4) applied to such an f is equivalent to the statement that

$$f(v) = \sum_{\alpha \varepsilon \mathscr{A}} d_\alpha\, tr\, \{ \hat{f}\,(\alpha)\, T^\alpha(v) \}\,,$$

the convergence being in the L^2 norm. Perhaps the most basic property of convolution is that, under Fourier transformation, it corresponds to

---

[1] It can be shown that the function of u whose value is $f(u)\, g\,(vu^{-1})$ is integrable (with respect to Haar measure) for almost all $v \,\varepsilon\, SO\,(n)$ and $f * g$ belongs to $L^1\,(SO(n))$. In fact $\|f * g\|_1 \leqq \|f\|_1\, \|g\|_1$. With the operation of convolution so defined, $L^1\,(SO(n))$ is a non-commutative Banach algebra.

pointwise multiplication. In the present situation this involves matrix multiplication and the precise formulation of this property is:

THEOREM (4.14). *If* (f*g) *denotes the Fourier transform of the convolution of the integrable functions* f *and* g *on* (SO (n) *then*

$$(f*g)\hat{}\;(\alpha) = \hat{f}\,(\alpha)\,\hat{g}\,(\alpha)$$

for all $\alpha \in \mathscr{A}$.

Proof. Using Fubini's theorem and the fact that T^α, being a representation, satisfies $T(v^{-1}) = T(u^{-1})\,T(uv^{-1})$ we have

$$(f*g)\hat{}\;(\alpha) = \int_{SO(n)} \Big\{ \int_{SO(n)} f(u)\,g(vu^{-1})\,du \Big\}\, T^\alpha(v^{-1})\,dv =$$

$$= \int_{SO(n)} f(u)\,T^\alpha(u^{-1}) \Big\{ \int_{SO(n)} g(vu^{-1})\,T^\alpha(uv^{-1})\,dv \Big\}\,du = \hat{f}\,(\alpha)\,\hat{g}\,(\alpha)$$

which proves the theorem.

This operation of convolution induces in a natural way a similar operation on functions defined on the surface of the unit sphere Σ_{n-1}. Suppose f and g are two such functions and let us assume that they are integrable with respect to Lebesgue measure on Σ_{n-1}. Then the functions $f^\#$ and $g^\#$, whose values at $v \in SO(n)$ are $f^\#(v) = f(v\mathbf{1})$ and $g^\#(v) = g(v\mathbf{1})$, belong to $L^1(SO(n))$ and

$$(f^\# * g^\#)(v) = \int_{SO(n)} f(w\mathbf{1})\,g(vw^{-1}\mathbf{1})\,dw = \int_{SO(n)} f(uw\mathbf{1})\,g(vw^{-1}u^{-1}\mathbf{1})\,dw =$$

$$= \int_{SO(n-1)} \Big\{ \int_{SO(n)} f(uw\mathbf{1})\,g(vw^{-1}\mathbf{1})\,dw \Big\}\,du\,.$$

Let $f^0(\xi) = \int_{SO(n-1)} f(u\xi)\,du$. If $v\mathbf{1} = \xi$ for $v \in SO(n)$ we put $t(v) = f^0(v\mathbf{1})$. The function t when satisfies $t(u_1 v u_2) = t(v)$ for all $u_1, u_2 \in SO(n-1)$. The fact that $t(vu_2) = t(v)$ for all $v \in SO(n)$ is obvious while, since the Haar measure of $SO(n-1)$ is translation invariant,

$$t(u_1 v) = f^0(u_1 v\mathbf{1}) = \int_{SO(n-1)} f(uu_1 v\mathbf{1})\,du = \int_{SO(n-1)} f(uv\mathbf{1})\,du = t(v)\,.$$

But, in the proof of theorem (4.11) we showed that a function t satisfying this property has the expansion (4.13). In view of (4.2) and (3.16), therefore, we see that f^0 depends only on $\xi\,.\,\mathbf{1}$. We shall write, therefore,

$$f_0(\xi.\mathbf{1}) = \int_{SO(n-1)} f(u\xi)\,du\,.$$

Thus,

$$(f^{\#} * g^{\#})(v) = \int_{SO(n)} g(vw^{-1}\mathbf{1})\{\int_{SO(n-1)} f(uw\mathbf{1})\,du\}\,dw$$

$$(4.15) \qquad = \int_{SO(n)} g(vw^{-1}\mathbf{1}) f_0(w\mathbf{1}.\mathbf{1})\,dw = \int_{SO(n)} g(w\mathbf{1}) f_0(w^{-1}v\mathbf{1}.\mathbf{1})\,dw =$$

$$= \int_{SO(n)} f_0(v\mathbf{1}.w\mathbf{1})\, g(w\mathbf{1})\,dw = \int_{\Sigma_{n-1}} f_0(\xi.\eta)\, g(\eta)\,d\eta\,.$$

This shows that the convolution of $f^{\#}$ and $g^{\#}$ depends on f_0 and g only (not on f, except in so far as f determines f_0).

Suppose $g = p$ is a spherical harmonic of degree k; that is, p belongs to $\mathcal{H}_n^{(k)}$. Then, by (2.17),

$$p(v\mathbf{1}) = \sum_{j=1}^{d_k} b_j\, t_{j1}^{(k)}(v)\,.$$

On the other hand, as we have just observed, $f_0(v\mathbf{1}.\mathbf{1})$ has the expansion (4.13):

$$f_0(v\mathbf{1}.\mathbf{1}) = \sum_{k=0}^{\infty} c_k\, t^{(k)}(v) = \sum_{k=0}^{\infty} c_k\, t_{11}^{(k)}(v)\,.$$

Moreover, (4.15) shows us that in calculating the convolution $f^{\#} * p^{\#}$ we can assume that $f^{\#}(w) = f_0(w\mathbf{1}.\mathbf{1})$. In this case,

$$\hat{f}^{\#}(\alpha) = \begin{pmatrix} c_k & 0 \ldots 0 \\ 0 & 0 \ldots 0 \\ \ldots & \ldots\ldots \\ 0 & 0 \ldots 0 \end{pmatrix} d_k^{-1}$$

when $\alpha = \alpha_k \in \mathcal{A}_1$ (see theorem (3.15)) and $\hat{f}(\alpha)$ is the zero matrix if $\alpha \in \mathcal{A} - \mathcal{A}_1$. Moreover, $\hat{p}^{\#}(\alpha)$ is the zero matrix if $\alpha \neq \alpha_k$ and

$$\hat{p}^{\#}(\alpha_k) = d_k^{-1} \begin{pmatrix} b_1 & b_2 \ldots b_{d_k} \\ 0 & 0 \ldots 0 \\ \ldots & \ldots\ldots \\ 0 & 0 \ldots 0 \end{pmatrix}.$$

Thus, by (4.14)

$$(f^{\#} * p^{\#})^{\wedge}(\alpha_k) = c_k\, d_k^{-2} \begin{pmatrix} b_1 & b_2 \ldots b_{d_k} \\ 0 & 0 \ldots 0 \\ \ldots & \ldots\ldots \\ 0 & 0 \ldots 0 \end{pmatrix} = d_k^{-1}\, c_k\, \hat{p}^{\#}(\alpha_k)$$

and $(f^{\#} * p^{\#})^{\wedge}(\alpha) = 0$ if $\alpha \neq \alpha_k$. Since the system $\{T^{\alpha}\}$ is complete it follows that $f^{\#} * p^{\#} = d_k^{-1} c_k p^{\#}$. This argument, in particular, proves the following classical result:

THEOREM (4.16). (Funk-Hecke theorem). *Suppose* p *is a spherical harmonic of degree* k *and* F *an integrable function on* $[-1, 1]$ *with respect to the measure* $(1-t^2)^{\frac{n-3}{2}}$ dt *then*

$$\int_{\Sigma_{n-1}} F(\xi . \eta)\, p(\eta)\, d\eta = \gamma_k p(\xi)$$

where,

$$\gamma_k^n = \gamma_k = a_k^{-2} c_n \int_{-1}^{1} F(t)\, P^{(k)}(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt .$$

Let us consider functions f on Σ_{n-1} that, like f^0, depend only on $\xi . \mathbf{1}$. That is, $f(u\xi) = f(\xi)$ for all $u \in SO(n-1)$ and $\xi \in \Sigma_{n-1}$. We have showed that if $f^{\#}$ is integrable then its Fourier transform is zero if T^{α} is not equivalent to any of the representations $S^{k,n}$ and

$$\hat{f}^{\#}(\alpha) = \gamma_k \begin{pmatrix} 10 \dots 0 \\ 00 \dots 0 \\ \dots\dots\dots \\ 00 \dots 0 \end{pmatrix}$$

when T^{α} is equivalent to $S^{k,n}$ (this was shown in the proof of (4.13) when the function is continuous. The more general result for integrable functions is an easy consequence of this more particular case). In this case we shall write

$$\hat{f}^{\#}(k) = \gamma_k ,$$

$k = 0, 1, 2, \dots$. That is, we identify α_k with k and the number γ_k with the matrix whose entry in the first column and first row is γ_k and having all other entries equal to zero. Thus, from the definition of the Fourier transform,

$$\hat{f}^{\#}(k) = \int_{SO(n)} f^{\#}(u)\, t^{(k)}(u^{-1})\, du = \int_{SO(n)} f^{\#}(u)\, t^{(k)}(u)\, du .$$

By theorem (3.1), equality (3.16) and the definition of $f^{\#}$, the last integral is equal to

$$a_k^{-2} \int_{\Sigma_{n-1}} f(\xi)\, Z_1(\xi)\, d\xi .$$

It is natural, therefore, to define the *Fourier transform* $\hat{f}$ of f by letting

$$(4.17) \qquad \hat{f}(k) = a_k^{-2} \int_{\Sigma_{n-1}} f(\xi)\, Z_1(\xi)\, d\xi = a_k^{-2} \int_{\Sigma_{n-1}} f(\xi)\, P^{(k)}(\xi.\mathbf{1})\, d\xi$$

for $k = 0, 1, 2, \ldots$.

If f and g are two such integrable functions, say $f(\xi) = F(\xi.\mathbf{1})$ and $g(\xi) = G(\xi.\mathbf{1})$ with

$$\int_{1}^{1} |F(t)|\,(1-t^2)^{\frac{n-3}{2}}\, dt < \infty \quad \textit{and} \quad \int_{-1}^{1} |G(t)|\,(1-t^2)^{\frac{n-3}{2}}\, dt < \infty ,$$

then, by (4.15),

$$(4.18) \qquad \hat{f}(k)\, \hat{g}(k) = [\int_{\Sigma_{n-1}} F(\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta]\, \hat{(k)} ,$$

$k = 0, 1, 2, \ldots$. From this we easily deduce that the algebra of this type of integrable functions on Σ_{n-1} with the convolution defined by

$$(4.19) \qquad \int_{\Sigma_{n-1}} F(\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta = (f*g)(\xi)$$

is a commutative Banach algebra. The fact $f*g$ is also a function that depends only on $\xi\,.\,\mathbf{1}$ is easily shown: if $u \in SO(n-1)$ then

$$\int_{\Sigma_{n-1}} F(u\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta = \int_{\Sigma_{n-1}} F(\xi.u^*\eta)\, G(\eta.u\mathbf{1})\, d\eta =$$

$$= \int_{\Sigma_{n-1}} F(\xi.u^*\eta)\, G(u^*\eta.\mathbf{1})\, d\eta = \int_{\Sigma_{n-1}} F(\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta .$$

That is, $(f*g)(u\xi) = (f*g)(\xi)$ for all $u \in SO(n-1)$ and $\xi \in \Sigma_{n-1}$.

If we left

$$\xi = (\xi_1, \ldots, \xi_{n-1}, \xi_n) , \quad \eta = (\eta_1, \ldots, \eta_{n-1}, \eta_n) ,$$

$$(1-\xi_n^2)^{1/2}\, \xi' = (\xi_1, \ldots, \xi_{n-1}) \quad \textit{and} \quad (1-\eta_n^2)^{1/2}\, \eta' = (\eta_1, \ldots, \eta_{n-1})$$

the integral in (4.19) becomes

$$\int_{\Sigma_{n-1}} F(\xi_n \eta_n + (1-\xi_n^2)^{1/2} (1-\eta_n^2)^{1/2}\, \xi' \,.\, \eta')\, G(\eta_n)\, d\eta .$$

In order to express the fact that this integral defines an operation on functions defined on $[-1, 1]$, we shall also denote it by $(F*G)(\xi_n) =$

$= (F^*G)(\xi.\mathbf{1})$. Putting $\eta_n = \cos\theta$, $0 \leqq \theta \leqq \pi$, $t = \xi' . \eta'$, $s = \xi_n$ and applying an argument similar to that used in the proof of (4.5) we obtain the fact that $(f^*g)(\xi)$ is equal to

$$(4.20) \qquad (F^*G)(s) =$$

$$c_{n-1} c_n \int_{-1}^{1} \int_{-1}^{1} F\left(sr + (1-s^2)^{1/2}(1-r^2)^{1/2} t\right) G(r) (1-r^2)^{\frac{n-3}{2}} (1-t^2)^{\frac{n-4}{2}} \, dr \, dt.$$

It follows from our discussion that this operation $(F, G) \to F * G$ is commutative and, with it, the linear space $L_n^1(-1, 1)$ of those functions F satisfying

$$|| F || = c_n \int_{-1}^{1} | F(t) | (1-t^2)^{\frac{n-3}{2}} dt < \infty$$

is a Banach algebra.

Formula (4.10) can now be given additional meaning. Putting $\xi_n = r$ and $\eta_n = s$ it becomes

$$(4.21) \qquad P^{(k)}(r) P^{(k)}(s) =$$

$$= a_k^2 c_{n-1} \int_{-1}^{1} P^{(k)}\left(rs + (1-r^2)^{1/2}(1-s^2)^{1/2} t\right) (1-t^2)^{\frac{n-4}{2}} dt .$$

If we define the Fourier transform of $F \in L_n^1(-1, 1)$ by letting

$$\hat{F}(k) = a_k^{-2} c_n \int_{-1}^{1} F(s) P^{(k)}(s) (1-s^2)^{\frac{n-3}{2}} ds$$

for $k = 0, 1, 2, \ldots$ (compare with (4.17)), formula (4.21) can be used to readily imply that

$$\hat{F}(k) \hat{G}(k) = [F^*G]^{\wedge}(k)$$

for $k = 0, 1, 2, \ldots$ (compare with (4.18)).¹)

---

¹) The convolution (4.20) was studied by Bochner; see *Proc. Nat. Acad. Sci. USA*, 40 (1954), 114-1147

§ 5. SPECIAL RESULTS FOR $n = 4$

In the literature (see in particular Bateman [1] Vol. 2 § 11.6) especially elegant formulas are given in the four dimensional case. These formulas can be obtained by using $SU(2)$ as a group acting transitively on Σ_3.

We begin by identifying $\mathbf{R}^4, \mathbf{C}^2, \mathbf{R}^+ \times SU(2) = \{ ru \colon r = a \text{ positive real, and } u \in SU(2) \}$, via the following maps.

$$x = (x_1, x_2, x_3, x_4) \leftrightarrow (x_1 + ix_2, x_3 + ix_4) = (\chi_1, \chi_2)$$

$$(\chi_1, \chi_2) \leftrightarrow -i \begin{pmatrix} -\bar{\chi}_2, & \chi_1 \\ \bar{\chi}_1, & \chi_2 \end{pmatrix} = i \, | x | \begin{pmatrix} -\bar{\chi}_2', & \chi_1' \\ \bar{\chi}_1', & \chi_2' \end{pmatrix} = | x | \, u_x$$

It is easily checked that

$$u_x = -i \begin{pmatrix} -\chi_2', & \chi_1' \\ \bar{\chi}_1', & \bar{\chi}_2' \end{pmatrix}$$

belongs to $SU(2)$.

Clearly, when $| x | = 1$ the correspondence $x \leftrightarrow u_x$ permits us to identify Σ_3 with $SU(2)$. We chose this map $x \to u_x$ in order to obtain (identifying $\mathbf{1}$ with (o, i)).

$$u_x \mathbf{1} = -i \begin{pmatrix} -\bar{\chi}_2, & \chi_1 \\ \bar{\chi}_1, & \chi_2 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = x .$$

If we consider the action of $SU(2)$ on itself obtained by left translation, this identification allows us to consider $SU(2)$ as a subgroup of $SO(4)$. That is, for $x \in \mathbf{R}^4$ and $u \in SU(2)$ we let $ux = | x | \, u \, u_x$. The mapping $x \to ux$ so defined is easily seen to be a rotation.

The normalized Lebesgue surface measure on Σ_3, being invariant under rotation, is actually the Haar measure on $SU(2)$.1)

In view of the Peter-Weyl theorem for $SU(2)$, a natural orthonormal basis for $L^2(SU(2)) = L^2(\Sigma_3)$ is obtained by considering the matrix entries of a complete system of irreducible representations.

We let $\mathcal{T}_u^{(k)}$ be the irreducible representation of $SU(2)$ realized on $\mathcal{P}^{(k)} = \mathcal{P}^{(k,2)}$, the space of homogeneous polynomials in $z = (z_1, z_2)$ of degree k, by letting

$$\mathcal{T}_u^{(k)} p(z) = p(u' z)$$

---

1) We have $\int_{SU(2)} f(u) \, du = \int_{\Sigma_3} f(u_\xi) \, d\xi$.

As can be seen from (2.6) an orthonormal basis of $\mathscr{P}^{(k)}$ is given by

$$p_j(z) = \binom{k}{j}^{1/2} z_1^j z_2^{k-j} \qquad j = 0, 1, \ldots, k\,.$$

We define the matrix entries of $\mathscr{T}_u^{(k)}$ with respect to this basis by

$$p_j(u'z) = \sum_{l=0}^{k} \tau_{lj}^{(k)}(u)\, p_l(z)$$

and we obtain associated functions on $\mathbf{R}^4$, which we also denote by $\tau_{lj}^{(k)}$, if we define

$$\tau_{lj}^{(k)}(x) = |\,x\,|^k\, \tau_{lj}^{(k)}(u_x)$$

We then have the following result:

THEOREM (5.1). *The representations* $\mathscr{T}^{(k)}$ *form a complete system of irreducible representations of* SU (2). *The functions* (k+1) $\tau_{lj}^{(k)}$ (x) *constitute an orthonormal basis of* $\mathscr{H}_4^{(k)}$ *with respect to the inner product introduced in* (3.9).

Proof. The completeness of $\mathscr{T}^{(k)}$ follows from the second part of the theorem and the completeness of spherical harmonics on Σ_3.

For $|\,x\,| = 1$ $\tau_{lj}^{(k)}(x) = \tau_{lj}^{(k)}(u_x)$; thus, the orthogonality relations follow from the Peter-Weyl theorem.

The dimension of $\mathscr{H}_4^{(k)}$ is $(k+1)^2$ so that it remains to show that the functions $\tau_{lj}^{(k)}(x)$ are actually homogeneous harmonic polynomials of degree k.

We have by the binomial formula

$$\sum_{j=0}^{k} \frac{1}{\binom{k}{j}}\, p_j(z)\, \overline{p_j(w)} = (z.w)^k\,.$$

hence

$$(u'z\cdot w)^k = \sum_{j=0}^{k} \frac{1}{\binom{k}{j}}\, p_j(u'z)\, \overline{p_j(w)} = \sum_{l,j=0}^{k} \frac{1}{\binom{k}{j}}\, \tau_{lj}^{(k)}(u)\, p_l(z)\, \overline{p_j(w)} \tag{5.2}$$

By the identification of $\mathbf{R}^4$ with $\mathbf{C}^2$ we have

$$\begin{aligned} |\,x\,|\, u_x'\, z\cdot w &= -i(-z_1\overline{w}_1\bar{\chi}_2 + z_2\overline{w}_1\bar{\chi}_1 + z_1\overline{w}_2\chi_1 + z_2\overline{w}_2\chi_2) = \\ &= -i\big[(z_2\overline{w}_1 + z_1\overline{w}_2)\,x_1 - i(z_2\overline{w}_1 + z_1\overline{w}_2)\,x_2 + (z_2\overline{w}_2 - z_1\overline{w}_1)\,x_3 + \\ &\qquad + i(z_2\overline{w}_2 + z_1\overline{w}_1)\,x_4\big] = \sum_{j=1}^{4} d_j x_j\,. \end{aligned} \tag{5.3}$$

Thus from (5, 2), (5, 3)

$$\sum_{l,j=0}^{k} \frac{1}{\binom{k}{j}} \tau_{lj}^{(k)}(x)\, p_l(z)\, \overline{p_j(w)} = \Big(\sum_{j=1}^{4} d_j x_j\Big)^k . \tag{5.4}$$

Each $\tau_{lj}^{(k)}(x)$ is a polynomial in $\mathscr{P}^k$. Moreover it is immediate that

$$\sum_{j=1}^{4} d_j^2 = 0 .$$

Thus

$$\Delta\Big(\sum_{j=1}^{4} d_j x_j\Big)^k = k(k-1)\Big(\sum_{j=1}^{4} d_j x_j\Big)^{k-2}\Big(\sum_{j=1}^{4} d_j^2\Big) = 0 .$$

This shows that

$$\tau_{lj}^{(k)}(x) \,\varepsilon\, \mathscr{H}_4^{(k)} .$$

We can now give an explicit formula for $\tau_{lj}^{(k)}(x)$:
Since

$$p_j(u_x' z) = \binom{k}{j}(-i)^k (z_2\bar{\chi}_1 - z_1\bar{\chi}_2)^j (z_1\chi_1 + z_2\chi_2)^{k-j} =$$

$$= \sum_{l=0}^{k} \tau_{lj}^{(k)}(x)\binom{k}{l} z_1^l z_2^{k-l} ,$$

letting $s = z_1/z_2$ we have

$$\binom{k}{j}(-i)^k(\bar{\chi}_1 - \bar{\chi}_2 s)^j(\bar{\chi}_1 s + \chi_2)^{k-j} = \sum_{l=0}^{k}\binom{k}{l}\tau_{lj}^{(k)}(x)\, s^l$$

Let

$$f(s) = \frac{\bar{x}_2}{|x|^2}(\chi_1 s + \chi_2),\quad 1 - f(s) = \frac{\chi_1}{|x|^2}(\bar{\chi}_1 - \bar{\chi}_2 s)$$

then

$$(-i)^k\binom{k}{j}[f(s)]^j[1-f(s)]^{k-j} = \frac{\bar{\chi}_2^j\bar{\chi}_1^{k-j}}{|x|^{2k}}\sum_{l=0}^{k}\binom{k}{l}\tau_{ij}^{k}(x)\, s^l$$

and using Taylor's formula for the l^{th} coefficient in this sum we obtain the classical Jacobi polynomial expression (see Bateman [1] Vol. 2 pp. 254)

$$\tau_{lj}^{(k)}(x) = \frac{(-i)^k\binom{k}{j}}{\binom{k}{l}\, l\,!}\,|x|^{2k}\,\overline{(\chi_2^j\chi_1^{k-j})}^{\,-1}\frac{d^l}{dt^l}\big[t^j(1-t)^{k-j}\big] ,$$

where

$$t = \frac{\chi_2 \bar{x}_2}{|x|^2} . ^{1)}$$

§ 6. THE FOURIER TRANSFORM OF FUNCTIONS ON $\mathbf{R}^n$

We have shown that $L^2(\Sigma_{n-1})$ can be decomposed into a direct sum of mutually orthogonal subspaces (the spaces $\mathcal{H}_n^{(k)}$) that are invariant and irreducible under the action of rotations. There exists a corresponding decomposition of $L^2(\mathbf{R}^n)$ and the spaces making up this decomposition are intimately connected with the Fourier transform of functions of n real variables. In this section we shall construct these spaces and study the action of the Fourier transform restricted to them. We shall see that also in this situation the rotation group $SO(n)$ and its representations play a central role.

If f belongs to $L^1(\mathbf{R}^n)$ its *Fourier transform* $\hat{f}$ is defined by letting

$$(\mathcal{F} f)(y) = \hat{f}(y) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x.y} dx$$

for $y \in \mathbf{R}^n$. 1)

Perhaps the simplest class of functions that is invariant under the action of the Fourier transform is the collection of *radial functions.* We recall that these are the functions on $\mathbf{R}^n$ that depend only on $|x|$; equivalently, f is radial if $\rho_v f = f$ for all $v \in SO(n)$, where the operator ρ_v is defined by

$$(\rho_v f)(x) = f(v^{-1} x)$$

for all $x \in \mathbf{R}^n$. Since Lebesgue measure is invariant under the action of rotations and $v = v^*$ when $v \in SO(n)$,

$$\int_{\mathbf{R}^n} f(x) e^{-2\pi i x.v^{-1}y} dx = \int_{\mathbf{R}^n} f(x) e^{-2\pi i vx.y} dx = \int_{\mathbf{R}^n} f(v^{-1} x) e^{-2\pi i x.y} dx .$$

That is,

$$(\mathcal{F} \rho_v) f = (\rho_v \mathcal{F}) f \tag{6.1}$$

---

1) It is not hard to use these results in order to obtain analogous results for $SO(3)$. We refer the reader to VILENKIN [11] for complete details.

1) When $f \varepsilon L^2(\mathbf{R}^n)$ the integral defining $\hat{f}$ is not defined in the Lebesgue sense. In this case, $\hat{f}$ is usually defined as the limit in the L^2 mean of the sequence $\hat{f}^k(y) = \int_{|x| \leqq k} f(x) e^{-2\pi i x \cdot y} dx$. In order to avoid technical difficulties that arise from this definition we shall restrict our attention to integrable functions

for all $f \in L^1(\mathbf{R}^n)$. This basic property, that Fourier transformation commutes with the action of rotations clearly implies.

THEOREM (6.2). *If* $f \in L^1(\mathbf{R}^n)$ *is radial then* $\hat{f}$ *is also a radial function.*

In order to extend this invariance property we introduce, for $k = 0, 1, 2, \ldots$, the class of functions $\mathfrak{h}^{(k)} = \mathfrak{h}_n^{(k)}$ mapping $\mathbf{R}^n$ into $\mathbf{C}^{d_k}$ having the form

$$F(x) = f(|x|)\left(Y_1(\xi), \ldots, Y_{d_k}(\xi)\right) = \left(F_1(x), \ldots, F_{d_k}(x)\right),$$

where

$$x = |x|\,\xi\,, \int_0^\infty f(r)\, r^{n-1}\, dr < \infty\ ^{1)} \text{ and } \{Y_1, Y_2, \ldots, Y_{d_k}\}$$

is an orthonormal basis of $\mathscr{H}_n^{(k)}$ such that $Y_1 = a_k^{-1} Z_1$ (that is, orthonormality is to be taken with respect to the inner product (2.6)). Such a basis was considered, for example, in theorem (2.16). When $k = 0$ this class is precisely the set of radial functions. It will be convenient if we choose the $Y_1, \ldots Y_{d_k}$ to be real-valued.

Let $T^{(k)} = (t_{lj}^{(k)})$ be the matrix of the representation $S^{k,n}$ with respect to the basis $\{Y_1, Y_2, \ldots, Y_{d_k}\}$; that is, the functions $t_{lj}^{(k)} = t_{lj}$ satisfy

$$(S_v^{k,n} Y_j)(\xi) = Y_j(v^{-1}\xi) = \sum_{l=1}^{d_k} t_{lj}(v)\, Y_l(\xi)$$

for $j = 1, 2, \ldots, d_k$. If we let

$$\rho_v F = (\rho_v F_1, \ldots, \rho_v F_{d_k})$$

we then have

$$(\rho_v F)(x) = f(|x|)\left(Y_1(v^{-1}\xi), \ldots, Y_{d_k}(v^{-1}\xi)\right) =$$

$$= f(|x|)\left(Y_1(\xi), \ldots, Y_{d_k}(\xi)\right)\begin{pmatrix} t_{11}(v) & \ldots & t_{1d_k}(v) \\ t_{21}(v) & \ldots & t_{2d_k}(v) \\ \ldots & \ldots & \ldots \\ t_{d_k1}(v) & \ldots & t_{d_kd_k}(v) \end{pmatrix} = (T_v^{(k)} F)(x)\,,$$

The last equality being the definition of the operator $T_v^{(k)}$ acting on F. That is,

$$\rho_v F = T_v^{(k)} F \tag{6.3}$$

---

1) This condition merely assures us that the radial function $g(x) = f(|x|)$ is integrable on $\mathbf{R}^n$.

for all $v \in SO(n)$. If we now apply the Fourier transform to each component of $\rho_v F$, it follows from (6.2) and (6.3) that

$$\rho_v \hat{F} = \rho_v(\hat{F}_1, \ldots, \hat{F}_{d_k}) = T_v^{(k)} \hat{F} . \tag{6.4}$$

The following, together with relation (6.4), shows that $\hat{F}$ must have the same form as F; that is,

$$\hat{F}(y) = \tilde{f}(|y|)(Y_1(\eta), \ldots, Y_{d_k}(\eta)) \tag{6.5}$$

for all $y = |y| \eta \in \mathbf{R}^n$.

THEOREM (6.6). *Suppose* $\mathrm{G} = (\mathrm{G}_1, \ldots, \mathrm{G}_{d_k})$ *is a continuous function mapping* $\mathbf{R}^n$ *into* $\mathbf{C}^{dk}$ *such that*

$$\rho_v G = T_v^{(k)} G \tag{6.7}$$

for all $\mathrm{v} \in \mathrm{SO}(\mathrm{n})$, *then*

$$G(y) = a_k^{-1} G_1(|y|\mathbf{1})(Y_1(\eta), \ldots, Y_{d_k}(\eta))$$

for all $\mathrm{y} = |\mathrm{y}|$ *in* $\mathbf{R}^n$.

Proof. Let $v \in SO(n)$ be such that $y = |y| v' \mathbf{1} = |y| v^{-1} \mathbf{1}$. Then, by (6.7)

$$G(y) = G(v^{-1}|y|\mathbf{1}) = (T_v^{(k)} G)(|y|\mathbf{1}) .$$

Consequently,

$$G_j(y) = \sum_{l=1}^{d_k} t_{lj}(v) G_l(|y|\mathbf{1}) \tag{6.8}$$

for $j = 1, 2, \ldots, d_k$. If $u \in SO(n-1)$ then $y = |y| v^{-1}\mathbf{1} = y = |y| v^{-1} u^{-1} \mathbf{1} = |y| (uv)^{-1} \mathbf{1}$; thus, if we replace v by uv in (6.8) we obtain

$$G_j(y) = \sum_{l=1}^{d_k} t_{lj}(uv) G_l(|y|\mathbf{1}) .$$

Integrating over $SO(n-1)$, therefore,

$$G_j(y) = \sum_{l=1}^{d_k} G_l(|y|\mathbf{1}) \int_{SO(n-1)} t_{lj}(uv)\, du .$$

But, by (3.2) and theorem (3.5) (or (3.15))

$$\int_{SO(n-1)} t_{lj}(uv)\,du = \begin{cases} t_{lj}(v) & when \quad l = 1 \\ 0 & when \quad l > 1 \end{cases}.$$

This equality and (2.17) show that

$$G_j(y) = G_j(v^{-1}|y|\mathbf{1}) = G_1(|y|\mathbf{1})\,\overline{Y_j(v^{-1}\mathbf{1})}\,a_k^{-1}\,.$$

(Since

$$\overline{t_{lj}(v)} = t_{jl}(v^{-1}) = \overline{Y_j(v^{-1}\mathbf{1})}\,)\,.$$

Writing $y = |y|\,\eta$, where $\eta = v^{-1}\,\mathbf{1}$, and using the fact that Y_j is real-valued, we obtain the desired result

$$G_j(y) = a_k^{-1}\,G_1(|y|\mathbf{1})\,Y_j(\eta)\,,$$

$j = 1, 2, ..., d_k$.

THEOREM (6.9). *Let* Y *be a spherical harmonic of degree* k *and* f *a function on* $(-\infty, \infty)$ *satisfying*

(*i*) $$\int_0^\infty |f(r)|\,r^{n-1}\,dr < \infty\,.$$

If $h(x) = f|x|)\,Y(\xi)$, *when* $x = |x|\,\xi \in \mathbf{R}^n$, *then* $h \in L^1(\mathbf{R}^n)$ *and*

$$\hat{h}(y) = \tilde{f}\,(|y|\;Y(\eta)$$

for all $y = |y|\,\eta \in \mathbf{R}^n$. *The transformation* $f \to \tilde{f}$ *depends only on* k *and* n *and, in particular, is independent of* $Y \in \mathscr{H}_n^{(k)}$.

Proof. Let $\{Y_1, ..., Y_{d_k}\}$ be the basis of $\mathscr{H}_n^{(k)}$ that was used in the previous theorem and $F(x) = (f(|x|)\,Y_1(\xi), ..., f(|x|)\,Y_{d_k}(\xi)) = (F_1(x), ..., F_{d_k}(x))$. Condition (i) guarantees that each of the functions $F_j, j = 1, ..., d_k$, is integrable.[1] Thus, $\hat{F} = (\hat{F}_1, ..., \hat{F}_{d_k})$ is well defined, continuous (as can be very easily shown), and satisfies relation (6.4). By theorem (6.6), therefore,

---

[1] Using polar coordinates $x = |x|\,\xi$, with $\xi \in \Sigma_{n-1}$, we have $\int_{\mathbf{R}^n} |F_j(x)|\,dx = \int_{\Sigma_{n-1}} \omega_{n-1}\,\{\int_0^\infty |f(r)|\,r^{n-1}\,dr\}\,|Y_j(\xi)|\,d\xi < \infty$, where ω_{n-1} is the "area" of Σ_{n-1}.

$$\hat{F}(y) = a_k^{-1} \hat{F}_1(|y|\mathbf{1})(Y_1(\eta), \dots, Y_{d_k}(\eta)) \quad \textit{for} \quad y = |y|\eta \in \mathbf{R}^n .$$

Putting $\tilde{f}(|y|) = a_k^{-1} \hat{F}_1(|y|\mathbf{1})$ we obtain equality (6.5). Since $\{Y_1, \dots, Y_{d_k}\}$ is a basis of $\mathcal{H}_n^{(k)}$ we can find coefficients $b_1, \dots, b_{d_k}$ such that

$$Y = \sum_{l=1}^{d_k} b_l Y_l .$$

Thus,

$$h(x) = \sum_{l=1}^{d_k} f(|x|) b_l Y_l(\xi) .$$

We have just shown that the Fourier transform of $F_l(x) = f(|x|) Y_l(\xi)$ has the values $\tilde{f}(|y|) Y_l(\eta)$. Thus,

$$\hat{h}(y) = \sum_{l=1}^{d_k} b_l \tilde{f}(|y|) Y_l(\eta) = \tilde{f}(|y|) \sum_{l=1}^{d_k} b_l Y_l(\eta) = \tilde{f}(|y|) Y(\eta) .$$

This proves the theorem.

It is not hard to give an explicit form for the mapping $f \to \tilde{f}$ in terms of the *Bessel functions*

$$J_\lambda(t) = \frac{(t/2)^\lambda}{\Gamma\left(\frac{2\lambda+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its}(1-s^2)^{\frac{2\lambda-1}{2}} ds .$$

We shall show, in fact, that

$$\text{(6.10)} \qquad \tilde{f}(t) = \gamma_{k,n} t^{\frac{2-n}{2}} \int_0^\infty f(r) J_{k+\frac{n-2}{2}}(2\pi t r) r^{n/2} dr . ^{1)}$$

Since $\tilde{f}$ is independent of $Y \in \mathcal{H}_n^{(k)}$ let us choose $h(x) = f(|x|) Z_1(\xi) = f(|x|) P^{(k)}(\xi.\mathbf{1})$. Then

$$\hat{f}(y) = \int_{\mathbf{R}^n} e^{-2\pi i y.x} f(|x|) P^{(k)}(\xi \cdot \mathbf{1}) dx =$$

---

1) We shall not calculate $\gamma_{k,n}$. The fact that this constant equals $2\pi i^{-k}$ can be shown by evaluating the integral in (6.10) when $f(r) = e^{-r^2}$ (see STEIN and WEISS [10], Chapter IV, section 3) or by using the constants obtained below.

$$= \omega_{n-1} \int_{0}^{\infty} r^{n-1} f(r) \{ \int_{\Sigma_{n-1}} e^{-2\pi i |y| r (\eta \cdot \xi)} P^{(k)} (\xi . \mathbf{1}) \, d\xi \} \, dr$$

Writing $y = t\,\eta$, this means that we have to compute

$$\int_{\Sigma_{n-1}} e^{-2\pi i r t (\eta \cdot \xi)} P^{(k)} (\xi . \mathbf{1}) \, d\xi \, .$$

But, by the Funk-Hecke theorem (4.16) this integral is equal to

$$P^{(k)} (\eta . \mathbf{1}) \, a_k^{-2} \, c_n \int_{-1}^{1} e^{-2\pi i r t s} P^{(k)} (s) (1 - s^2)^{\frac{n-3}{2}} \, ds \, .$$

On the other hand, by (4.4), and, then integrating by parts k times we have

$$\int_{-1}^{1} e^{-2\pi i r t s} P^{(k)} (s) (1 - s^2)^{\frac{n-3}{2}} \, ds = \alpha_{k,n} \int_{-}^{1} e^{-2\pi i r t s} \left[\frac{d^k}{dt^k} (1 - s^2)^{k + \frac{n-3}{2}} \right] ds$$

$$= \beta_{k,n} \int_{-} (rt)^k \, e^{2\pi i r t s} (1 - s^2)^{k + \frac{n-3}{2}} \, ds \, .$$

The last integral, however, is the one involved in the definition of J_λ when $\lambda = (2k+n-2)/2$. Equality (6.10) now follows immediately.^{1})

BIBLIOGRAPHY

[1] Bateman, H., *Bateman Manuscript Project*, Vol. 1 and 2, N. Y. (1953).

[2] Calderón, A. P., *Integrales Singulares y sus Aplicaciones a Ecuaciones Diferenciales Hiperbólicas*. Fasc. 3, Cursos y Seminarios de Matematica, Univ. de Buenos Aires (1959).

[3] —— and A. Zygmund, Singular Integral Operators and Differential Equations. *Am. J. of Math.*, Vol. LXXIX, No. 4 (1957), pp. 901-921.

[4] Cartan, E., *Œuvres Complètes*. Gauthier-Villars, Paris (1939).

[5] Godemont, R., A Theory of Spherical Functions, I. *Trans. Am. Math. Soc.*, *73* (1952), pp. 496-556.

[6] Dieudonné, J. *Representacion de Groupos Compactos y Funciones Esfericas*. Fasc. 14, Cursos y Seminarios de Matematica, Univ. de Buenos Aires (1964).

---

1) The Bessel functions we have encountered here arise in much the same way as did the ultraspherical Polynomials. Instead of the group *SO* (*n*), however, one must study the group of all rigid motions on $\mathbf{R}^{(n)}$ (see Vilenkin [11] for details).

[7] PONTRIAGIN, L., *Topological Groups.* 2nd Edition, Moscow (1957).
[8] PUKANSZKY, L., *Representation of Groups.* Dunod, Paris (1968).
[9] SEELEY, R. T., Spherical Harmonics, No. 11 of the H. Ellsworth Slaught Memorial Papers. *Am. Math. Monthly*, Vol. 73, No. 4 (1966), pp. 115-121.
[10] STEIN, E. M. and G. WEISS, *Fourier Analysis in Euclidean spaces.* Princeton Univ. Press, N. J. (1969).
[11] VILENKIN, N., *Special Functions and the Theory of Group Representations.* Moscow (1965).
[12] WEYL, H. *The Theory of Groups and Quantum Mechanics*, 2nd Ed., Dover (1931).

Washington University
St. Louis, Mo.

(Reçu le 31 juillet 1968)