§4. SOME PROPERTIES OF SPHERICAL HARMONICS

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Theorem (3.15) shows us how the spherical harmonics we introduced in § 2 can be obtained from the general theory of representations of compact groups applied to $SO(n)$. We have also obtained several properties of these spherical harmonics by using simple arguments based on this general theory. We claim that essentially all the well-known classical facts concerning these special functions can be obtained by equally simple arguments. In the next section we justify this claim by deriving a number of important results in the theory of spherical harmonics. Our arguments will again be based on the general theory of representations of compact groups.

§ 4. Some properties of spherical harmonics

The zonal harmonics Z_1 are often expressed in terms of certain polynomial functions $P_n^{(k)}$ restricted to the interval $[-1,1]$ that are called the *ultra spherical* (or Gegenbauer) polynomials. We have already obtained such an expression in § 2. In fact let

$$P^{(k)}(t) = a_k^2\Big(t^k + \sum_{1 \leqq j \leqq k/2} (-1)^j \frac{\beta_1\beta_2 \ldots \beta_j}{\alpha_1\alpha_2 \ldots \alpha_j} t^{k-2j}(1-t^2)^j\Big) \tag{4.1}$$

for $-1 \leqq t \leqq 1, \alpha_j = 2j(2j+n-3)$, $\beta_j = (k-2j+1)(k-2j+2)$ and $a_k^2 = Z_1(\mathbf{1})$. If $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \Sigma_{n-1}$ and we put $t = \xi_n$, so that $1 - t^2 = \xi_1^2 + \ldots + \xi_{n-1}^2$, the expression in parenthesis becomes the polynomial (2.14) exaluated at ξ. The observation we made in the paragraph following the proof of Corollary (2.15) is equivalent to the fact $Z_1(\xi)$ and $P^{(k)}(t)$ are equal. Writing $t = \xi \cdot \mathbf{1}$ this equality becomes

$$Z_1(\xi) = P^{(k)}(\xi \cdot \mathbf{1}). \tag{4.2}$$

Usually, the ultraspherical polynomials are introduced in one of two ways. One method is to apply the Gram-Scmidt process to the powers $1, t, t^2, \ldots$ restricted to the interval $[-1, 1]$ with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(t)\overline{g(t)}(1-t^2)^{\frac{n-3}{2}} dt. \tag{4.3}$$

Another definition of the polynomials $P^{(k)}$ involves the k^{th} derivative of $(1-t^2)^{(2k+n-3)/2}$:

$$P^{(k)}(t) = \alpha_{k,n}(1-t^2)^{(3-n)/2} \frac{d^k}{dt^k}(1-t^2)^{(n+2k-3)/2} \tag{4.4}$$

It is not hard to show that the definition (4.1) is equivalent to these two definitions. One way of doing this is by first establishing the following lemma:

LEMMA (4.5). *Suppose φ is a continuous function on* $[-1, 1]$ *then*

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot\eta)\, d\xi = c_n \int_{-1}^{1} \varphi(t)(1-t^2)^{\frac{n-3}{2}}\, dt$$

where

$$c_n^{-1} = \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}}\, dt\,.$$

Proof. This lemma is really of a geometrical nature. First, we note that

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot\eta)\, d\xi$$

is independent of η since, if u is a rotation,

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot u\eta)\, d\xi = \int_{\Sigma_{n-1}} \varphi(u^*\xi\cdot\eta)\, d\xi = \int_{\Sigma_{n-1}} \varphi(\xi\cdot\eta)\, d\xi\,.$$

Thus, we can choose $\eta = \mathbf{1}$. Having done this, we can then evaluate the integral of $\varphi(\xi.\mathbf{1})$ over Σ_{n-1} by first integrating over a *parallel perpendicular to* $\mathbf{1}$, $\sigma_\theta = \{\xi \in \Sigma_{n-1} : \xi\,.\,\mathbf{1} = \cos\theta\}$, $0 \leqq \theta \leqq \pi$, and then integrating the function of θ we have obtained over the interval $[0, \pi]$. Since $\varphi(\xi.\mathbf{1}) = \varphi(\cos\theta)$ is constant over this parallel and the Lebesgue measure of σ_θ is $\omega_{n-2}(\sin\theta)^{n-2}$ (where ω_{n-2} is the measure of the surface, Σ_{n-2}, of the unit sphere of $\mathbf{R}^{n-1}$) we must have

$$\int_{\Sigma_{n-1}} \varphi(\xi\cdot\mathbf{1})\, d\xi = \tilde{c}_n \int_0^\pi \omega_{n-2}\, \varphi(\cos\theta)(\sin\theta)^{n-2}\, d\theta\,.$$

The constant

$$\tilde{c}_n = 1 \Big/ \int_{-1}^{1} \omega_{n-2}(\sin\theta)^{n-2}\, d\theta$$

must be introduced since we normalized $d\xi$ so that

$$\int_{\Sigma_{n-1}} d\xi = 1\,.$$

The lemma now follows from the change of variables $t = \cos\theta$.

One of the assertions of theorem (3.15) is that

$$< p, q > \; = \int_{\Sigma_{n-1}} p(\xi)\overline{q(\xi)}\, d\xi = 0 \quad \textit{if} \quad p \in \mathscr{H}_n^{(k)} \textit{ and } q \in \mathscr{H}_n^{(j)} .$$

If we apply this result to $p(\xi) = Z_1^{(k)}(\xi)$ and $q(\xi) = Z_1^{(j)}(\xi)$, (4.2) and lemma (4.5) then imply

$$\int_{-1}^{1} P^{(k)}(t)\, P^{(j)}(t)\,(1-t^2)^{\frac{n-3}{2}}\, dt = 0 \tag{4.6}$$

when $k \neq j$. Since $P^{(k)}$ is a polynomial of degree k, for $k = 0, 1, 2, \ldots$ we have the following result:

THEOREM (4.7). *The polynomials* $P^{(k)}(t)$, $k = 0, 1, 2, \ldots$, *form a complete orthogonal system in* $L^2(-1, 1)$ *with respect to the inner product* (4.3).

Let

$$Q(t) = (1-t^2)^{(3-n)/2} \frac{d^k}{dt^k} (1-t^2)^{(n+2k-3)/2}$$

and $R(t)$ a polynomial of degree $\leqq (k-1)$. Then, integrating by parts k times, we obtain

$$\int_{-1}^{1} R(t)\, Q(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt = \int_{-1}^{1} R(t) \frac{d^k}{dt^k} (1-t^2)^{(n+2k-3)/2}\, dt = 0 .$$

In particular, Q is orthogonal to $P^{(j)}$ for $j = 0, 1, \ldots, k-1$. Since $Q(t)$ is of degree k it follows from theorem (4.7) that there must exist a constant $\alpha = \alpha_{k,n}$ such that $P^{(k)}(t) = \alpha_{k,n}\, Q(t)$. This is precisely equality (4.4).

The following result, a useful tool in the theory of singular integrals and partial differential equations (see Calderon and Zygmund [3] and Seeley [9]), is an immediate application of the relation (3.17) between the inner products $< , >$ and $(,)$.

THEOREM (4.8). *If* p *is a harmonic polynomial on* $\mathbf{R}^n$ *that is homogeneous of degree* k *then there exists a constant* $B = B(\|\alpha\|, n)$, *depending only on the dimension* n *and* $\|\alpha\|$, *such that*

$$\int_{\Sigma_{n-1}} | D^\alpha p(\xi) |^2\, d\xi \leqq B (k+1)^{2\|\alpha\|} \int_{\Sigma_{n-1}} | p(\xi) |^2\, d\xi .$$

Proof. Suppose $p(x) = \sum_{\|\beta\|=k} c_\beta x^\beta$. Since, by assumption, $p \in \mathcal{H}_n^{(k)}$ It follows that any one of its partial derivatives, say $\frac{\partial p}{\partial x_n}$, belongs to $\mathcal{H}_n^{(k-1)}$. If Σ' denotes summation over all $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that $\|\beta\| = k$ and $\beta_n > 0$, then

$$\frac{\partial p}{\partial x_n}(x) = \sum{}' \beta_n c_\beta x^{\beta-1}.$$

Thus, by (2.6″),

$$\left(\frac{\partial p}{\partial x_n}, \frac{\partial p}{\partial x_n}\right)_{(k-1)} = \sum_{\|\beta\|=k}{}' \frac{\beta!}{(k-1)!} \beta_n |c_\beta|^2.$$

Since $\|\beta\| = k$ implies $\beta_n \leqq k$ it follows that

$$(p, p)_{(k)} = \sum_{\|\beta\|=k} \frac{\beta!}{k!} |c_\beta|^2 \geqq \sum_{\|\beta\|=k}{}' \frac{\beta!}{(k-1)!} \beta_n \frac{1}{\beta_n k} |c_\beta|^2$$

$$\geqq \frac{1}{k^2} \sum_{\|\beta\|=k}{}' \frac{\beta!}{(k-1)!} \beta_n |c_\beta|^2 = \frac{1}{k^2}\left(\frac{\partial p}{\partial x_n}, \frac{\partial p}{\partial x_n}\right)_{(k-1)}.$$

Repeating this argument we obtain

$$(4.9) \qquad (D^\alpha p, D^\alpha p)_{(k-\|\alpha\|)} \leqq \left[\frac{k!}{(k-\|\alpha\|)!}\right]^2 (p, p)_{(k)}.$$

From, (3.17), (4.9), (2.19) [1]) and (3.19) we then have

$$\int_{\Sigma_{n-1}} |D^\alpha p(\xi)|^2 d\xi = \langle D^\alpha p, D^\alpha p \rangle = A_{k-\|\alpha\|} (D^\alpha p, D^\alpha p)_{(k-\|\alpha\|)}$$

$$\leqq A_{k-\|\alpha\|} \left[\frac{k!}{(k-\|\alpha\|)!}\right]^2 (p, p)_{(k)} =$$

$$= A_{k-\|\alpha\|} \left[\frac{k!}{(k-\|\alpha\|)!}\right]^2 A_k^{-1} \langle p, p \rangle = C(\alpha, n, k) \int_{\Sigma_{n-1}} |p(\xi)|^2 d\xi.$$

Here

$$C(\alpha, n, k) = A_{k-\|\alpha\|} \left[\frac{k!}{(k-\|\alpha\|)!}\right]^2 A_k^{-1} =$$

[1]) We only announced equality (2.19) and have not proved it. The reader can check that its proof is particularly easy when the dimension n is even. Equality (2.18), which was proved, can be used here to obtain essentially the same estimates.

$$= \left(\frac{a_{k-||\alpha||}}{a_k}\right)^2 \left(\frac{d_k}{d_{k-||\alpha||}}\right) \frac{(k!)^2}{[(k-||\alpha||)\,!]^2} =$$

$$= \left\{\prod_{j=k-||\alpha||}^{k-1} \frac{2j+n-2}{j+n-2}\right\} \left\{\frac{(k+n-3)\,!\,(2k+n-2)\,(k-||\alpha||)\,!}{k\,!\,(k-||\alpha||+n-3)\,!\,(2k-2||\alpha||+n-2)}\right\}$$

$$\frac{(k!)^2}{[(k-||\alpha||)\,!]^2}$$

The first product in brackets consists of $||\alpha|| - 1$ terms, each less than or equal to 2; therefore, it is dominated by $2^{||\alpha||-1}$. The second bracket times the last fraction reduce to

$$\frac{2k+n-2}{2k-2||\alpha||+n-2} \left\{\frac{k\,!\,(k+n-3)\,!}{(k-||\alpha||)\,!\,(k-||\alpha||+n-3)\,!}\right\}.$$

Since $||\alpha|| \leqq k$ and $3 \leqq n$,

$$\frac{2k+n-2}{2k-2||\alpha||+n-2} \leqq \frac{2||\alpha||+n-2}{n-2} = \frac{2||\alpha||}{n-2} + 1 \leqq 2||\alpha|| + 1 .$$

The term in brackets, however, consists of the product of $||\alpha||$ numbers $(k+n-3)\,(k+n-4)\,...\,(k+n-||\alpha||-2)$ times another product, $k\,(k-1)...(k-||\alpha||+1)$, of $||\alpha||$ numbers. Since each factor is no larger than $k+n-3$, the term in brackets is dominated by

$$(k+n-3)^{2||\alpha||} \leqq (n-2)^{2||\alpha||}\,(k+1)^{2||\alpha||}.$$

Thus,

$$C(\alpha, n, k) \leqq 2^{||\alpha||-1}\,(2||\alpha||+1)\,(n-2)^{2||\alpha||}\,(k+1)^{2||\alpha||}$$

and the theorem is proved with $B\,(||\alpha||, n) = 2^{||\alpha||-1}\,(2||\alpha||+1)\,(n-2)^{2||\alpha||}$.

Many classical formulae are easily derived from the general theory we have developed. For example, let us consider the relation

$$P^{(k)}(\xi_n)\,P^{(k)}(\eta_n) = \tag{4.10}$$

$$= a_k^2\,c_{n-1} \int_{-1}^{1} P^{(k)}\,(\xi_n\eta_n + (1-\xi_n^2)^{1/2}\,(1-\eta_n^2)^{1/2}\,t)\,(1-t^2)^{\frac{n-4}{2}}\,dt$$

which we shall show to be true for all $\xi = (\xi_1, \xi_2, ..., \xi_n)$ and

$$\eta = (\eta_1, \eta_2, ..., \eta_n) \text{ in } \Sigma_{n-1}$$

(the constant c_{n-1} was introduced in the statement of lemma (4.5)). This is formula (20) on page 177 of the Bateman Manuscript Project [1], Volume 1.

Equality (4.10) can be regarded as a functional equation defining the zonal harmonics Z_1 or the ultraspherical polynomials $P^{(k)}$ (in the same sense that the relation $f(x+y) = f(x)f(y)$ can be regarded as a functional equation defining the exponential functions).

We claim that (4.10), as well as the statement in the last paragraph, are nothing but a transcription of the following theorem:

THEOREM (4.11). *Let* $\mathrm{t}^{(k)}$, $\mathrm{k} = 0, 1, 2, \ldots$, *be the function defined by* (3.16). *Then,*

(i)
$$t^{(k)}(u_1 v u_2) = t^{(k)}(v)$$

for $\mathrm{u}_1, \mathrm{u}_2$ *in* SO (n−1) *and* v *in* SO (n). *Moreover,*

(ii)
$$\int_{SO(n-1)} t^{(k)}(v_1 u v_2)\, du = t^{(k)}(v_1)\, t^{(k)}(v_2)$$

for all $\mathrm{v}_1, \mathrm{v}_2 \in$ SO (n).

Conversely, suppose t *is a continuous function on* SO (n) *that is not identically zero which satisfies*

$$t(u_1 v u_2) = t(v)$$

for $\mathrm{u}_1, \mathrm{u}_2$ *in* SO (n−1, v *in* SO (n) *and*

$$\int_{SO(n-1)} t(v_1 u v_2)\, du = t(v_1)\, t(v_2)$$

for all $\mathrm{v}_1, \mathrm{v}_2$ *in* SO (n). *Then there exists a non-negative integer* k *such that* $\mathrm{t} = \mathrm{t}^{(k)}$.

Before proving theorem (4.11) we show that equality (ii) does imply (4.10). In fact, from (2.17) and (4.2)

$$a_k^2\, t^{(k)}(u) = P^{(k)}(u\mathbf{1}.\mathbf{1})$$

for all $u \in SO(n)$ (recall that $t^{(k)}$ and, therefore, Z_1 are real valued. This was shown immediately preceding (3.7)). Thus, (4.11), part (ii), becomes

$$a_k^{-2} \int_{SO(n-1)} P^{(k)}(v_1 u v_2 \mathbf{1}.\mathbf{1})\, du = a_k^{-4} P^{(k)}(v_1 \mathbf{1}.\mathbf{1})\, P^{(k)}(v_2 \mathbf{1}.\mathbf{1})$$

If we put $v_2\, \mathbf{1} = \xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $v_1^*\, \mathbf{1} = v_1'\, \mathbf{1} = \eta = (\eta_1, \eta_2, \ldots, \eta_n)$, then $v_2\, \mathbf{1}\,.\,\mathbf{1} = \xi_n$ and $v_1\, \mathbf{1}\,.\,\mathbf{1} = \mathbf{1}\,.\,v_1^*\mathbf{1} = \eta_n$.

Hence,

$$a_k^2 \int_{SO(n-1)} P^{(k)}(u\xi.\eta)\,du = P^{(k)}(\xi_n)\,P^{(k)}(\eta_n)\,. \tag{4.12}$$

We now write

$$\xi = (1-\xi_n^2)^{1/2}\,\xi' + \xi_n\,\mathbf{1} \quad and \quad \eta = (1-\eta_n^2)^{1/2}\,\eta' + \eta_n\,\mathbf{1}\,,$$

where

$$\xi' = (\xi_1', \xi_2', \ldots, \xi_{n-1}', 0) \quad and \quad \eta' = (\eta_1', \eta_2', \ldots, \eta_{n-1}', 0)$$

belong to Σ_{n-1} and are orthogonal to $\mathbf{1}$ (clearly,

$$\xi_j' = \xi_j\,/\,(1-\xi_n^2)^{1/2} \quad and \quad \eta_j' = \eta_j\,/\,(1-\eta_n^2)^{1/2}$$

when ξ_n and η_n are not ± 1; in which case, $\xi_j' = 0 = \eta_j'$ for $1 \leqq j \leqq n-1$). We shall also denote $(\xi_1', \xi_2', \ldots, \xi_{n-1}')$ and $(\eta_1', \eta_2', \ldots, \eta_{n-1}')$ by ξ' and η'; that is, we identify Σ_{n-2} with those points of Σ_{n-1} having last coordinate 0. Thus, for u in $SO\,(n-1)$

$$u\,\xi\,.\,\eta = (1-\xi_n^2)^{1/2}\,(1-\eta_n^2)^{1/2}\,u\,\xi'\,.\,\eta' + \xi_n\,\eta_n\,.$$

An application of theorem (3.1) and lemma (4.5), therefore, gives us

$$\int_{SO(n-1)} P^{(k)}(u\xi.\eta)\,du = \int_{\Sigma_{n-2}} P^{(k)}\big((1-\xi_n^2)^{1/2}(1-\eta_n^2)^{1/2}\,\xi'\,.\,\eta' + \xi_n\,\eta_n\big)\,d\,\xi =$$

$$= c_{n-1} \int_{-1}^{1} P^{(k)}\big((1-\xi_n^2)^{1/2}(1-\eta_n^2)^{1/2}\,t + \xi_n\,\eta_n\big)\,(1-t^2)^{\frac{n-4}{2}}\,dt\,.$$

Equality (4.10) now follows from this last one and (4.12).

We now turn to the proof of theorem (4.11). Since $tr\,\{AB\} = tr\,\{BA\}$ for any two matrices A and B, we have $\chi_k\,(u_1\,v u_2\,u) = \chi_k\,(v u_2\,u u_1)$. Hence, since the Haar measure of $SO\,(n-1)$ is both left and right invariant,

$$t^{(k)}(v) = \int_{SO(n-1)} \chi_k(vu)\,du = \int_{SO(n-1)} \chi_k(vu_2uu_1)\,du = \int_{SO(n-1)} \chi_k(u_1vu_2u)\,du =$$

$$= t^{(k)}\,(u_1 v u_2)\,.$$

This establishes (i). In order to show (ii) we choose a matrix valued representation equivalent to $S^{k,n}$ in such a way that $t_{11}\,(v) = t^{(k)}\,(v)$ for $v \in SO\,(n)$. We can do this, for example, by choosing an orthonormal basis of $\mathscr{H}_n^{(k)}$ whose first element is $a_k^{-1}\,Z_1$ (see the discussion preceeding (3.7)). Then, by (1.2),

$$\int_{SO(n-1)} t^{(k)}(v_1uv_2)\,du = \sum_{l=1}^{d_k} \int_{SO(n-1)} t_{1l}(v_1)\,t_{l1}\,(uv_2)\,du$$

and, by (3.2),

$$\int_{SO(n-1)} t_{l1}(uv_2)\,du = \begin{cases} t_{11}(v_2) & if\ l = 1 \\ 0 & if\ 1 < l \leqq d_k \end{cases}.$$

We therefore obtain the desired result

$$\int_{SO(n-1)} t^{(k)}(v_1uv_2)\,du = t_{11}(v_1)\,t_{11}(v_2) = t^{(k)}(v_1)\,t^{(k)}(v_2).$$

We now show the converse. Since $t(vu) = t(v)$ for all $v \in SO(n)$ and $u \in SO(n-1)$ it follows from (3.4) and theorem (3.15) that

$$t(v) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} c_l^{(k)}\, t_{l1}^{(k)}(v),$$

the convergence being in $L^2(SO(n))$ (the $t_{ij}^{(k)}$ *'s* are the entries of the matrix valued representation equivalent to $S^{k,n}$ that we chose when we established equality (ii). On the other hand, the fact that $t(uv) = t(v)$ for all $v \in SO(n)$ and $u \in SO(n-1)$ implies that $c_l^{(k)} = 0$ for $l \neq 1$, since we can apply the same argument that was used in order to establish (3.4) by allowing the first row of $(t_{ij}^{(k)})$ to assume the role that was played by the first column.[1]) Thus,

$$t(v) = \sum_{k=0}^{\infty} c_1^{(k)}\, t_{11}^{(k)}(v) = \sum_{k=0}^{\infty} c_k\, t^{(k)}(v), \tag{4.13}$$

the convergence being in $L^2(SO(n))$. Suppose $c_{k_0} \neq 0$ for some k_0. Then

$$d_k \int_{SO(n)} t^{(k_0)}(v) \{ \int_{SO(n-1)} t(vuw)\,du \}\,dv =$$

$$d_k \int_{SO(n-1)} \{ \int_{SO(n)} t^{(k_0)}(vw^{-1}u^{-1})\,t(v)\,dv \}\,du =$$

$$d_k \int_{SO(n-1)} t^{(k_0)}(vw^{-1})\,t(v)\,dv = d_k \int_{SO(n)} t^{(k_0)}(vuw^{-1})\,t(vu)\,dv =$$

$$d_k \int_{SO(n-1)} \{ \int_{SO(n)} t^{(k_0)}(vuw^{-1})\,t(vu)\,dv \}\,du =$$

$$d_k \int_{SO(n)} \{ \int_{SO(n-1)} t^{(k_0)}(vuw^{-1})\,du \}\,t(v)\,dv =$$

$$d_k \int_{SO(n)} t^{(k_0)}(v)\,t^{(k_0)}(w^{-1})\,t(v)\,dv = d_k^{-1}\,c_{k_0}\,t^{(k_0)}(w^{-1}) = d_k^{-1}\,c_{k_0}\,t^{(k_0)}(w)$$

(recall that $t^{(k)}$ is real valued and, thus, $t^{(k)}(w^{-1}) = \bar{t}^{(k)}(w) = t^{(k)}(w)$).

[1]) The reader can verify that this is the case by replacing $T(vu)$ by $T(uv)$ in equality (3.3).

On the other hand,

$$d_k \int_{SO(n)} t^{(k_0)}(v) \{ \int_{SO(n-1)} t(vuw)\, du \}\, dv =$$

$$= d_k \int_{SO(n)} t^{(k_0)}(v)\, t(v)\, t(w)\, dv = d_k^{-1}\, c_{k_0}\, t(w)\,.$$

Consequently,

$$c_{k_0}\, t(w) = c_{k_0}\, t^{(k_0)}(w)\,. \quad \text{Since} \quad c_{k_0} \neq 0$$

this implies $t = t^{(k_o)}$
and theorem (4.11) is proved.

The fact that relation (4.10) can be regarded as a functional equation defining the zonal harmonics is not its only significance. The general methods we used in establishing it are connected with the operation of convolution in $L^1(SO(n))$, the space of integrable functions on $SO(n)$. Suppose f, g belong to this space, then their *convolution* $f * g$ is defined by letting

$$(f * g)(v) = \int_{SO(n)} f(u)\, g(vu^{-1})\, du$$

for all $v \in SO(n)$.[1])

Let $\{T^\alpha\}$, $\alpha \in \mathscr{A}$, be a complete system of irreducible matrix valued representations of $SO(n)$. For $f \in L^1(SO(n))$ we then define its (matrix valued) *Fourier transform* (or its system of *Fourier coefficients*) by putting

$$\hat{f}(\alpha) = \int_{SO(n)} f(u)\, T^\alpha(u^{-1})\, du$$

for $\alpha \in \mathscr{A}$. If f is also square integrable this definition is consistent with the Fourier coefficients introduced in the first section. In fact, it can be easily shown that Corollary (1.4) applied to such an f is equivalent to the statement that

$$f(v) = \sum_{\alpha \varepsilon \mathscr{A}} d_\alpha\, tr \{\hat{f}(\alpha)\, T^\alpha(v)\}\,,$$

the convergence being in the L^2 norm. Perhaps the most basic property of convolution is that, under Fourier transformation, it corresponds to

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1) It can be shown that the function of u whose value is $f(u)\, g(vu^{-1})$ is integrable (with respect to Haar measure) for almost all $v \varepsilon SO(n)$ and $f * g$ belongs to $L^1(SO(n))$. In fact $\|f * g\|_1 \leqq \|f\|_1 \|g\|_1$. With the operation of convolution so defined, $L^1(SO(n))$ is a non-commutative Banach algebra.

pointwise multiplication. In the present situation this involves matrix multiplication and the precise formulation of this property is:

THEOREM (4.14). *If* (f*g) *denotes the Fourier transform of the convolution of the integrable functions* f *and* g *on* (SO (n) *then*

$$\widehat{(f*g)}\,(\alpha) = \hat{f}\,(\alpha)\,\hat{g}\,(\alpha)$$

for all $\alpha \in \mathscr{A}$.

Proof. Using Fubini's theorem and the fact that T^{α}, being a representation, satisfies $T(v^{-1}) = T(u^{-1})\,T(uv^{-1})$ we have

$$\widehat{(f*g)}\,(\alpha) = \int_{SO(n)} \{\int_{SO(n)} f(u)\,g(vu^{-1})\,du\}\,T^{\alpha}(v^{-1})\,dv =$$

$$= \int_{SO(n)} f(u)\,T^{\alpha}(u^{-1})\{\int_{SO(n)} g(vu^{-1})\,T^{\alpha}(uv^{-1})\,dv\}\,du = \hat{f}\,(\alpha)\,\hat{g}\,(\alpha)$$

which proves the theorem.

This operation of convolution induces in a natural way a similar operation on functions defined on the surface of the unit sphere Σ_{n-1}. Suppose f and g are two such functions and let us assume that they are integrable with respect to Lebesgue measure on Σ_{n-1}. Then the functions $f^{\#}$ and $g^{\#}$, whose values at $v \in SO(n)$ are $f^{\#}(v) = f(v\mathbf{1})$ and $g^{\#}(v) = g(v\mathbf{1})$, belong to $L^1(SO(n))$ and

$$(f^{\#}*g^{\#})(v) = \int_{SO(n)} f(w\mathbf{1})\,g(vw^{-1}\mathbf{1})\,dw = \int_{SO(n)} f(uw\mathbf{1})\,g(vw^{-1}u^{-1}\mathbf{1})\,dw =$$

$$= \int_{SO(n-1)} \{\int_{SO(n)} f(uw\mathbf{1})\,g(vw^{-1}\mathbf{1})\,dw\}\,du\,.$$

Let $f^0(\xi) = \int_{SO(n-1)} f(u\xi)\,du$. If $v\mathbf{1} = \xi$ for $v \in SO(n)$ we put $t(v) = f^0(v\mathbf{1})$. The function t when satisfies $t(u_1 v u_2) = t(v)$ for all $u_1, u_2 \in SO(n-1)$. The fact that $t(vu_2) = t(v)$ for all $v \in SO(n)$ is obvious while, since the Haar measure of $SO(n-1)$ is translation invariant,

$$t(u_1 v) = f^0(u_1 v\mathbf{1}) = \int_{SO(n-1)} f(uu_1 v\mathbf{1})\,du = \int_{SO(n-1)} f(uv\mathbf{1})\,du = t(v)\,.$$

But, in the proof of theorem (4.11) we showed that a function t satisfying this property has the expansion (4.13). In view of (4.2) and (3.16), therefore, we see that f^0 depends only on $\xi\,.\,\mathbf{1}$. We shall write, therefore,

$$f_0(\xi.\mathbf{1}) = \int_{SO(n-1)} f(u\xi)\,du\,.$$

Thus,

$$(f^{\#} * g^{\#})(v) = \int_{SO(n)} g(vw^{-1}\mathbf{1}) \{ \int_{SO(n-1)} f(uw\mathbf{1})\, du \}\, dw$$

$$(4.15) \qquad = \int_{SO(n)} g(vw^{-1}\mathbf{1}) f_0(w\mathbf{1}.\mathbf{1})\, dw = \int_{SO(n)} g(w\mathbf{1}) f_0(w^{-1}v\mathbf{1}.\mathbf{1})\, dw =$$

$$= \int_{SO(n)} f_0(v\mathbf{1}.w\mathbf{1})\, g(w\mathbf{1})\, dw = \int_{\Sigma_{n-1}} f_0(\xi.\eta)\, g(\eta)\, d\eta\,.$$

This shows that the convolution of $f^{\#}$ and $g^{\#}$ depends on f_0 and g only (not on f, except in so far as f determines f_0).

Suppose $g = p$ is a spherical harmonic of degree k; that is, p belongs to $\mathscr{H}_n^{(k)}$. Then, by (2.17),

$$p(v\mathbf{1}) = \sum_{j=1}^{d_k} b_j\, t_{j1}^{(k)}(v)\,.$$

On the other hand, as we have just observed, $f_0(v\mathbf{1}.\mathbf{1})$ has the expansion (4.13):

$$f_0(v\mathbf{1}.\mathbf{1}) = \sum_{k=0}^{\infty} c_k\, t^{(k)}(v) = \sum_{k=0}^{\infty} c_k\, t_{11}^{(k)}(v)\,.$$

Moreover, (4.15) shows us that in calculating the convolution $f^{\#} * p^{\#}$ we can assume that $f^{\#}(w) = f_0(w\mathbf{1}.\mathbf{1})$. In this case,

$$\hat{f}^{\#}(\alpha) = \begin{pmatrix} c_k & 0 \dots 0 \\ 0 & 0 \dots 0 \\ \dots & \dots \\ 0 & 0 \dots 0 \end{pmatrix} d_k^{-1}$$

when $\alpha = \alpha_k \in \mathscr{A}_1$ (see theorem (3.15)) and $\hat{f}(\alpha)$ is the zero matrix if $\alpha \in \mathscr{A} - \mathscr{A}_1$. Moreover, $\hat{p}^{\#}(\alpha)$ is the zero matrix if $\alpha \neq \alpha_k$ and

$$\hat{p}^{\#}(\alpha_k) = d_k^{-1} \begin{pmatrix} b_1 & b_2 \dots b_{d_k} \\ 0 & 0 \dots 0 \\ \dots & \dots \\ 0 & 0 \dots 0 \end{pmatrix}.$$

Thus, by (4.14)

$$(f^{\#} * p^{\#})\,\hat{}\,(\alpha_k) = c_k\, d_k^{-2} \begin{pmatrix} b_1 & b_2 \dots b_{d_k} \\ 0 & 0 \dots 0 \\ \dots & \dots \\ 0 & 0 \dots 0 \end{pmatrix} = d_k^{-1}\, c_k\, \hat{p}^{\#}(\alpha_k)$$

and $(f^{\#} * p^{\#})^{\wedge}(\alpha) = 0$ if $\alpha \neq \alpha_k$. Since the system $\{T^{\alpha}\}$ is complete it follows that $f^{\#} * p^{\#} = d_k^{-1} c_k p^{\#}$. This argument, in particular, proves the following classical result:

THEOREM (4.16). (Funk-Hecke theorem). *Suppose* p *is a spherical harmonic of degree* k *and* F *an integrable function on* $[-1, 1]$ *with respect to the measure* $(1-t^2)^{\frac{n-3}{2}}\,dt$ *then*

$$\int_{\Sigma_{n-1}} F(\xi.\eta)\, p(\eta)\, d\eta = \gamma_k p(\xi)$$

where,

$$\gamma_k^n = \gamma_k = a_k^{-2} c_n \int_{-1}^{1} F(t)\, P^{(k)}(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt\,.$$

Let us consider functions f on Σ_{n-1} that, like f^0, depend only on $\xi\,.\,\mathbf{1}$. That is, $f(u\xi) = f(\xi)$ for all $u \in SO(n-1)$ and $\xi \in \Sigma_{n-1}$. We have showed that if $f^{\#}$ is integrable then its Fourier transform is zero if T^{α} is not equivalent to any of the representations $S^{k,n}$ and

$$\hat{f}^{\#}(\alpha) = \gamma_k \begin{pmatrix} 10 \ldots 0 \\ 00 \ldots 0 \\ \ldots\ldots\ldots \\ 00 \ldots 0 \end{pmatrix}$$

when T^{α} is equivalent to $S^{k,n}$ (this was shown in the proof of (4.13) when the function is continuous. The more general result for integrable functions is an easy consequence of this more particular case). In this case we shall write

$$\hat{f}^{\#}(k) = \gamma_k\,,$$

$k = 0, 1, 2, \ldots$. That is, we identify α_k with k and the number γ_k with the matrix whose entry in the first column and first row is γ_k and having all other entries equal to zero. Thus, from the definition of the Fourier transform,

$$\hat{f}^{\#}(k) = \int_{SO(n)} f^{\#}(u)\, t^{(k)}(u^{-1})\, du = \int_{SO(n)} f^{\#}(u)\, t^{(k)}(u)\, du\,.$$

By theorem (3.1), equality (3.16) and the definition of $f^{\#}$, the last integral is equal to

$$a_k^{-2} \int_{\Sigma_{n-1}} f(\xi)\, Z_1(\xi)\, d\xi\,.$$

It is natural, therefore, to define the *Fourier transform* $\hat{f}$ of f by letting

$$(4.17)\qquad \hat{f}(k) = a_k^{-2} \int_{\Sigma_{n-1}} f(\xi)\, Z_1(\xi)\, d\xi = a_k^{-2} \int_{\Sigma_{n-1}} f(\xi)\, P^{(k)}(\xi.\mathbf{1})\, d\xi$$

for $k = 0, 1, 2, \dots$.

If f and g are two such integrable functions, say $f(\xi) = F(\xi.\mathbf{1})$ and $g(\xi) = G(\xi.\mathbf{1})$ with

$$\int_{1}^{1} |F(t)|\,(1-t^2)^{\frac{n-3}{2}}\, dt < \infty \quad \textit{and} \quad \int_{-1}^{1} |G(t)|\,(1-t^2)^{\frac{n-3}{2}}\, dt < \infty ,$$

then, by (4.15),

$$(4.18)\qquad \hat{f}(k)\, \hat{g}(k) = [\int_{\Sigma_{n-1}} F(\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta]\,\hat{}\,(k) ,$$

$k = 0, 1, 2, \dots$. From this we easily deduce that the algebra of this type of integrable functions on Σ_{n-1} with the convolution defined by

$$(4.19)\qquad \int_{\Sigma_{n-1}} F(\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta = (f*g)(\xi)$$

is a commutative Banach algebra. The fact $f*g$ is also a function that depends only on $\xi\,.\,\mathbf{1}$ is easily shown: if $u \in SO(n-1)$ then

$$\int_{\Sigma_{n-1}} F(u\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta = \int_{\Sigma_{n-1}} F(\xi.u^*\eta)\, G(\eta.u\mathbf{1})\, d\eta =$$

$$= \int_{\Sigma_{n-1}} F(\xi.u^*\eta)\, G(u^*\eta.\mathbf{1})\, d\eta = \int_{\Sigma_{n-1}} F(\xi.\eta)\, G(\eta.\mathbf{1})\, d\eta .$$

That is, $(f*g)(u\xi) = (f*g)(\xi)$ for all $u \in SO(n-1)$ and $\xi \in \Sigma_{n-1}$.

If we left

$$\xi = (\xi_1, \dots, \xi_{n-1}, \xi_n) , \quad \eta = (\eta_1, \dots, \eta_{n-1}, \eta_n) ,$$

$$(1-\xi_n^2)^{1/2}\, \xi' = (\xi_1, \dots, \xi_{n-1}) \quad \textit{and} \quad (1-\eta_n^2)^{1/2}\, \eta' = (\eta_1, \dots, \eta_{n-1})$$

the integral in (4.19) becomes

$$\int_{\Sigma_{n-1}} F(\xi_n \eta_n + (1-\xi_n^2)^{1/2} (1-\eta_n^2)^{1/2}\, \xi' . \eta')\, G(\eta_n)\, d\eta .$$

In order to express the fact that this integral defines an operation on functions defined on $[-1, 1]$, we shall also denote it by $(F*G)(\xi_n) =$

$= (F^*G)(\xi.\mathbf{1})$. Putting $\eta_n = \cos\theta$, $0 \leqq \theta \leqq \pi$, $t = \xi' . \eta'$, $s = \xi_n$ and applying an argument similar to that used in the proof of (4.5) we obtain the fact that $(f^*g)(\xi)$ is equal to

(4.20)
$$(F^*G)(s) =$$

$$c_{n-1}\, c_n \int_{-1}^{1}\int_{-1}^{1} F\big(sr + (1-s^2)^{1/2}(1-r^2)^{1/2}\, t\big)\, G(r)\,(1-r^2)^{\frac{n-3}{2}}(1-t^2)^{\frac{n-4}{2}}\, dr\, dt.$$

It follows from our discussion that this operation $(F, G) \to F * G$ is commutative and, with it, the linear space $L_n^1(-1, 1)$ of those functions F satisfying

$$\| F \| = c_n \int_{-1}^{1} | F(t) |\, (1-t^2)^{\frac{n-3}{2}}\, dt < \infty$$

is a Banach algebra.

Formula (4.10) can now be given additional meaning. Putting $\xi_n = r$ and $\eta_n = s$ it becomes

(4.21)
$$P^{(k)}(r)\, P^{(k)}(s) =$$

$$= a_k^2\, c_{n-1} \int_{-1}^{1} P^{(k)}\big(rs + (1-r^2)^{1/2}(1-s^2)^{1/2}\, t\big)(1-t^2)^{\frac{n-4}{2}}\, dt\,.$$

If we define the Fourier transform of $F \in L_n^1(-1, 1)$ by letting

$$\hat{F}(k) = a_k^{-2}\, c_n \int_{-1}^{1} F(s)\, P^{(k)}(s)\,(1-s^2)^{\frac{n-3}{2}}\, ds$$

for $k = 0, 1, 2, \ldots$ (compare with (4.17)), formula (4.21) can be used to readily imply that

$$\hat{F}(k)\, \hat{G}(k) = \widehat{[F^*G]}\,(k)$$

for $k = 0, 1, 2, \ldots$ (compare with (4.18)).[1])

[1]) The convolution (4.20) was studied by Bochner; see *Proc. Nat. Acad. Sci. USA*, 40 (1954), 114-1147