2. Proof of Theorem 1

SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES

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To the memory of J. Karamata

1. INTRODUCTION

We will give simple proofs of the following uniqueness theorems on minimal surfaces:

THEOREM 1 (Bernstein). *Let* z = f (x, y) *be a minimal surface in euclidean three-space defined for all* x, y. *Then* f (x, y) *is a linear function.*

THEOREM 2. *A closed minimal surface of genus zero on the three-sphere must be totally geodesic and is hence a great sphere.*

Theorem 2 has been proved by Almgren [1] and Calabi [2].

2. PROOF OF THEOREM 1

Let

$$W = \left(1 + f_x^2 + f_y^2\right)^{\frac{1}{2}} \geqq 1 . \tag{1}$$

The proof is based on the identity

$$\Delta \log\left(1 + \frac{1}{W}\right) = K , \tag{2}$$

where Δ is the Laplacian relative to the induced riemannian metric of the minimal surface M and K is its Gaussian curvature.

Suppose (2) be true. Let ds be the element of arc on M. Introduce the conformal metric

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$$d\sigma = \left(1 + \frac{1}{W}\right) ds\,. \tag{3}$$

If p, q are isothermal coordinates on M, so that

$$ds^2 = \lambda^2 (dp^2 + dq^2)\,, \tag{4}$$

we have

$$K = -\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right) \log \lambda\,, \tag{5}$$

$$\Delta = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right).$$

Applying this to the metric $d\sigma$, we find immediately that its gaussian curvature is zero, or that the metric is flat.

On the other hand, it is clear that

$$ds \leqq d\sigma \leqq 2\,ds\,. \tag{6}$$

It follows that the metric $d\sigma$ on M is complete, for it dominates ds and ds is complete. We have therefore on M a complete flat riemannian metric $d\sigma$. By a well-known theorem, M, with the metric $d\sigma$, is isometric to the (ξ, η)-plane with its standard flat metric, i.e.,

$$d\sigma^2 = d\xi^2 + d\eta^2\,. \tag{7}$$

Since $K \leqq 0$, we have, from (2) and (5),

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \log \left(1 + \frac{1}{W}\right) \leqq 0\,. \tag{8}$$

The function $\log\left(1 + \frac{1}{W}\right)$, considered as a function in the (ξ, η)-plane, is therefore superharmonic. It is also clearly non-negative. By a well-known theorem on superharmonic functions ([3], p. 130) it must be a constant. Equation (2) then gives $K = 0$, which implies that M is a plane.

The proof of (2) is a standard calculation. It will be proved at the end of § 4 as a special case of a more general formula.

An advantage of this proof is the fact that, unlike many other known proofs, complex function theory is not used.