3. Proof of Theorem 2

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Let S^3 be the unit sphere in the euclidean 4-space E^4. By an orthonormal frame in E^4 is meant an ordered set of vectors e_α, $0 \leqq \alpha \leqq 3$, satisfying

$$(9) \qquad (e_\alpha, e_\beta) = \delta_{\alpha\beta}, \qquad 0 \leqq \alpha, \beta, \gamma \leqq 3,$$

where the left-hand side is the scalar product of the vectors in question. The space of all orthonormal frames can be identified with the group $SO(4)$. We introduce in $SO(4)$ the Maurer-Cartan forms $\omega_{\alpha\beta}$ according to the equations

$$(10) \qquad de_\alpha = \sum_\beta \omega_{\alpha\beta} e_\beta$$

or

$$(11) \qquad \omega_{\alpha\beta} = (de_\alpha, e_\beta).$$

It follows from (9) that

$$(12) \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0.$$

Exterior differentiation of (10) gives the Maurer-Cartan structure equations of $SO(4)$, which are

$$(13) \qquad d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}.$$

There is a fibering

$$(14) \qquad SO(4) \to S^3 = SO(4) / SO(3),$$

with the projection defined by sending the frame $e_0\, e_1\, e_2\, e_3$ to the unit vector e_0.

Suppose a smooth surface

$$(15) \qquad M \to S^3$$

be described by the vector e_0. We restrict to frames such that e_3 is the unit normal vector to M at e_0. There are two choices for e_3, any one of which is called an orientation of M. Suppose M be oriented. Then the frames are defined up to a rotation of the vectors e_1, e_2 in the tangent plane. In other words, our restricted family of frames is a circle bundle over M, for which the structure equations (13) are valid.

The condition that e_3 is a normal vector at e_0 implies

(16) $$\omega_{03} = 0 .$$

Taking its exterior derivative and using (13), we get

$$\omega_{01} \wedge \omega_{13} + \omega_{02} \wedge \omega_{23} = 0 .$$

Since M is an immersed surface, we have $\omega_{01} \wedge \omega_{02} \neq 0$ and Cartan's lemma allows us to set

(17) $$\omega_{13} = a\omega_{01} + b\omega_{02} , \qquad \omega_{23} = b\omega_{01} + c\omega_{02} .$$

The condition for a minimal surface is the vanishing of the mean curvature:

(18) $$a + c = 0 .$$

Let

(19) $$\alpha = \omega_{01} + i\omega_{02} , \qquad \beta = \omega_{13} + i\omega_{23} .$$

The structure equations (13) give

$$d\alpha = - i\omega_{12} \wedge \alpha ,$$

(20) $$d\beta = - i\omega_{12} \wedge \beta .$$

Under a rotation of e_1 e_2 both α and β will be multiplied by the same complex number of absolute value 1. It follows that

(21) $$\alpha \wedge \bar{\beta} , \qquad \alpha\bar{\beta} ,$$

which are exterior and ordinary two-forms respectively, are globally defined on our oriented surface M.

Suppose from now on that M is a minimal surface. Condition (18) can be written

(22) $$\beta = A\bar{\alpha} , \qquad A = a + ib .$$

In this case the first form in (21) vanishes identically, while

(23) $$\alpha\bar{\beta} = \bar{A}\alpha^2 .$$

Taking the exterior derivative of (22) and using (20), we get

(24) $$dA + 2iA\omega_{12} \equiv 0 , \qquad \text{mod } \bar{\alpha} .$$

The induced riemannian metric on M has an underlying complex structure which makes M into a Riemann surface. We wish to show that the form in (23) is a quadratic differential in the sense of Riemann surfaces. For this purpose let z be a local complex coordinate on M, so that

$$\alpha = \lambda dz . \tag{25}$$

Then we have, locally,

$$\alpha\beta = (\lambda^2 \bar{A})\, dz^2 .$$

Exterior differentiation of (25) and use of (20) give

$$d\lambda + i\lambda\omega_{12} \equiv 0 , \qquad \mathrm{mod}\; dz .$$

Combining with (24), we get

$$\frac{\partial}{\partial \bar{z}} (\lambda^2 \bar{A}) = 0$$

i.e., the coefficient of dz^2 in $\alpha\beta$ is holomorphic.

Since M is of genus zero, the quadratic differential must vanish. This implies $A = 0$ and that M is a great sphere.

The proof given above is not essentially different from those of Almgren and Calabi. The main idea of using the quadratic differential in surface theory goes back to H. Hopf. The formalism developed in this proof should also be useful in the study of other problems on surfaces in S^3.

4. A FORMULA ON NON-PARAMETRIC MINIMAL HYPERSURFACES IN EUCLIDEAN SPACE

Instead of proving formula (2) we will establish a more general formula for a non-parametric minimal hypersurface in the euclidean $(n+1)$-space E^{n+1}, which seems to have an independent interest.

Suppose $x: M \to E^{n+1}$ be an immersion of an n-dimensional manifold M in E^{n+1}. We consider orthonormal frames $x\, e_1 \ldots e_{n+1}$ in E^{n+1}, such that $x \in M$ and e_{n+1} is the unit normal vector to M at x. We have then

$$\begin{aligned} dx &= \sum_i \omega_i\, e_i , \\ de_i &= \sum_k \omega_{ik}\, e_k + \omega_{i,n+1}\, e_{n+1} , \qquad 1 \leqq i, j, k, l \leqq n , \\ de_{n+1} &= - \sum_i \omega_{i,n+1}\, e_i , \end{aligned} \tag{26}$$