

# 6. Application to Tauberian theorems

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$$\begin{aligned}
 (5.10) \quad & \lim \frac{\mathcal{J}_V(t) - \mathcal{J}_V(\lambda t)}{[\mathcal{J}_U(t) - \mathcal{J}_U(\lambda t)] L(t)} = \\
 & = \lim \frac{L(t) \mathcal{J}_U(t) - L(\lambda t) \mathcal{J}_U(\lambda t)}{L(t) \mathcal{J}_U(t) - L(\lambda t) \mathcal{J}_U(\lambda t)} = 1.
 \end{aligned}$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t) L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t) L(t)}$$

and so (5.1) is true.

## 6. APPLICATION TO TAUBERIAN THEOREMS

If the measure  $U$  varies regularly at infinity, then its Laplace transform  $\omega$  varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any  $\alpha \geq 0$  and slowly varying function  $L$  the two relations

$$(6.1) \quad U(x) \sim x^\alpha L(x) \quad \omega(\lambda) \sim \Gamma(\alpha + 1) \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here  $x \rightarrow \infty$  but  $\lambda \rightarrow 0$ . [The sign  $\sim$  indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \quad U(x) = \int_0^x y^p F(dy)$$

is the truncated  $p^{\text{th}}$  moment of a probability distribution  $F$  on the positive half axis. For simplicity let  $p$  stand for a positive integer. Then  $U_p(x) = 1 - F(x)$  and  $\omega = (-1)^p \phi^{(p)}$  where  $\phi$  is the Laplace-Stieltjes transform of  $F$ . If  $\omega$  varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \quad 1 - F(x) \sim \frac{\alpha}{p - \alpha} x^{\alpha - p} L(x) \quad \text{when} \quad \alpha < p$$

$$(6.3b) \quad 1 - F(x) = o(x^\alpha L(x)) \quad \text{when} \quad \alpha = p.$$

(Note that necessarily  $0 \leq \alpha \leq p$  because the measure  $F$  is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail  $1 - F(x)$ , and vice versa.

From the continuity theorem for Laplace transforms one concludes without difficulties that *dominated variation of  $U$  at  $\infty$  is equivalent to dominated variation of  $\omega$  at 0*. Even in the case of mere dominated variation the behavior of  $\omega = (-1)^p \phi^{(p)}$  therefore permits inferences concerning the behavior of the tail  $1 - F$ , but naturally the conclusions will lack the pleasing precision of (6.3). It is therefore remarkable that precise asymptotic equivalence relations can be obtained when comparing two probability distributions  $F$  and  $G$  with Laplace transforms  $\phi$  and  $\gamma$ .

A typical *Tauberian ratio limit theorem* would state that the two relations

$$(6.4) \quad \gamma^{(p)}(\lambda) \sim \phi^{(p)}(\lambda) L\left(\frac{1}{\lambda}\right) \quad \lambda \rightarrow 0$$

and

$$(6.5) \quad 1 - G(x) \sim [1 - F(x)] L(x) \quad x \rightarrow \infty$$

*imply each other*. This is not true in full generality; indeed, (6.3b) points to exceptional situations even when the transforms in (6.4) vary regularly. However, our results yield a variety of fairly general sufficient conditions for the validity of the conclusion. Suppose, for example, that for some constants  $A$  and  $\alpha < p$

$$(6.6) \quad (-1)^p \phi^{(p)}(\lambda) < A \lambda^{-\alpha}$$

for  $\lambda$  sufficiently small. It is easily seen in this case that  $U$  varies dominatedly with exponent  $\alpha$  and (6.4) is equivalent to

$$(6.7) \quad V \leftrightarrow UL$$

in the sense of (5.1). (Here  $V$  stands for the truncated moment function of  $G$  defined as in (6.2).) Theorem 4 then asserts that (5.3) holds, and this implies

$$(6.8) \quad V_p \leftrightarrow U_p L$$

whenever  $U$  is bounded away from 0. Now (6.5) differs from (6.8) only notationally, and we know that the condition (4.1) guarantees that  $R_U$  is bounded away from 0 and that  $U_p = 1 - F$  varies dominatedly. Again, (4.1) holds if, and only if, each limit of a convergent sequence of measures  $U(t_n dx)/U(t_n)$  attributes a positive measure to  $(0, \infty)$ . This requirement is satisfied if, and only if,

$$(6.9) \quad \liminf_{\varepsilon \rightarrow 0} \frac{\phi^{(p)}(\varepsilon \lambda_0)}{\phi^{(p)}(\varepsilon)} < 1$$

for some  $\lambda_0 > 1$ . Accordingly, if the conditions (6.6) and (6.9) hold then (6.4) implies (6.6.) as well as the dominated variation of  $1 - F$  and  $1 - G$ .

Our results permit various paraphrases of the sufficient conditions, and also of the ratio limit theorem itself. That (6.6) by itself is not sufficient is shown by (6.3b); without (6.9) certain subsequences may exhibit the pattern of slow variation, and the conclusion (6.5) must be replaced by a weaker conclusion of the form (6.3b).

## 7. ON THE TAILS OF INFINITELY DIVISIBLE DISTRIBUTIONS

To illustrate the usefulness of the notion of dominated variation in probabilistic contexts we prove the following

**PROPOSITION.** *Let  $H$  stand for an infinitely divisible probability distribution with Lévy measure  $M \{dx\}$ . If  $M$  varies dominatedly at  $+\infty$  then*

$$(7.1) \quad 1 - H(x) \sim M \{ (x, \infty) \}, \quad x \rightarrow +\infty$$

in the sense that the ratio of the two sides tends to unity at all points of continuity. (A very special case involving regular variation is mentioned in [1], p. 540.)

**PROOF.** We shall show that the general proposition follows easily from the special case where  $M$  is supported by the positive half axis and has a finite mass  $\mu$ . In this case

$$(7.2) \quad M \{ (x, \infty) \} = \mu [1 - F(x)] \quad x > 0$$

where  $F$  is a probability distribution on  $(0, x)$ , and  $H$  reduces to the compound Poisson distribution given by

$$(7.3) \quad H(x) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} F^{n*}(x) \quad x > 0.$$

We proceed to prove the assertion (7.1) for distributions of this form assuming that  $1 - F$  varies dominatedly. Note that  $F^{n*}$  is the distribution of the sum  $S_n = X_1 + \dots + X_n$  of  $n$  mutually independent random variables with the common distribution  $F$ . Since these variables are positive, the event  $\{S_n > x\}$  occurs whenever at least one among the  $n$  variables exceeds  $x$ , and so

$$(7.4) \quad 1 - F^{n*}(x) \geq n [1 - F(x)] - \binom{n}{2} [1 - F(x)]^2$$