ON SOME GENERALISATIONS OF ABEL SUMMABILITY

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B. Kuttner

To the memory of J. Karamata

1. With the usual terminology, a sequence $\{s_n\}$ is described as Abel summable to s if

$$(1-x)\sum_{n=0}^{\infty} s_n x^n$$

converges for $0<x<1$, and tends to s as $x\to 1-$. For our present purposes, it is convenient to put $x = t/(1+t)$; thus the definition takes the form that

$$\phi(t) = \frac{1}{1+t}\sum_{n=0}^{\infty} s_n \left(\frac{t}{1+t}\right)^n \tag{1}$$

converges for $t>0$, and tends to s as $t\to\infty$. A generalisation which has been considered by Kogbetliantz [3] and Lord [4] is to replace (1) by

$$\phi(\alpha; t) = \frac{1}{1+t}\sum_{n=0}^{\infty} s_n^{\alpha} \left(\frac{t}{1+t}\right)^n, \tag{2}$$

where $\alpha>-1$ and where $\{s_n^{\alpha}\}$ is the (C, α) mean of $\{s_n\}$; that is to say

$$s_n^{\alpha} = \frac{1}{\binom{n+\alpha}{n}}\sum_{\nu=0}^{n}\binom{n-\nu+\alpha-1}{n-\nu} s_\nu .$$

Here we write, as usual,

$$\binom{m+\beta}{m} = \frac{(m+\beta)(m+\beta-1)\ldots(1+\beta)}{m!}.$$

If (2) converges for all $t>0$, and if $\phi(\alpha; t)\to s$ as $t\to\infty$, we say that $\{s_n\}$ is summable (A, α) to s. It is easily seen that if, for a given $\alpha>-1$, (2) converges for all $t>0$ then the same thing holds for any other $\alpha>-1$. It is known that, if this holds, then, for $\beta>\alpha>-1$,

$$\phi(\beta; t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha+1)}\, t^{-\beta} \int_0^t (t-u)^{\beta-\alpha-1} u^{\alpha} \phi(\alpha; u)\, du \,. \tag{3}$$

As is known, it follows easily from (3) that, for $\alpha > -1$, summability (A, α) increases in strength with increasing α; that is to say, if $\beta > \alpha > -1$ and if $\{s_n\}$ is summable (A, α) to s, then it is also summable (A, β) to s.

A different generalisation has been introduced by Borwein [1]. For $\lambda > -1$, let

$$\phi_\lambda(t) = \frac{1}{(1+t)^{\lambda+1}} \sum_{n=0}^{\infty} \binom{n+\lambda}{n} s_n \left(\frac{t}{1+t}\right)^n . \tag{4}$$

If (4) converges for all $t > 0$ and if $\phi_\lambda(t) \to s$ as $t \to \infty$, we say that $\{s_n\}$ is summable A_λ to s. It is again clear that if, for a given $\lambda > -1$, (4) converges for all $t > 0$, then the same thing holds for any other $\lambda > -1$. Borwein has shown that, if this holds, then, for $\lambda > \mu > -1$,

$$\phi_\mu(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\,\Gamma(\mu+1)}\, t^{-\lambda} \int_0^t (t-u)^{\lambda-\mu-1} u^{\mu} \phi_\lambda(u)\, du \,. \tag{5}$$

Using (5), Borwein proved that, for $\lambda > -1$, summability A_λ increases in strength with decreasing λ.

Let us now combine these two ideas. For $\alpha > -1$, $\lambda > -1$, let

$$\phi_\lambda(\alpha; t) = \frac{1}{(1+t)^{\lambda+1}} \sum_{n=0}^{\infty} \binom{n+\lambda}{n} s_n^{\alpha} \left(\frac{t}{1+t}\right)^n . \tag{6}$$

If (6) converges for all $t > 0$, and if $\phi_\lambda(\alpha; t) \to s$ as $t \to \infty$, we say that $\{s_n\}$ is summable (A_λ, α) to s. The object of this paper is to compare the strengths of (A_λ, α) for different values of α, λ. We will show that (assuming that $\alpha > -1$, $\lambda > -1$) the strength of (A_λ, α) depends only on the value of $\alpha - \lambda$; further, the method increases in strength with increasing $\alpha - \lambda$. In other words, we have the following result.

THEOREM. *Suppose that $\alpha > -1, \lambda > -1, \beta > -1, \mu > -1, \beta - \mu \geqslant \alpha - \lambda$. If $\{s_n\}$ is summable (A_λ, α) to s, then it is summable (A_β, μ) to s.*

We remark that this theorem clearly includes the result that if $\beta - \mu = \alpha - \lambda$ then summabilities (A_λ, α), (A_μ, β) are equivalent.

2. In order to prove the theorem, we make use of the idea of the Hausdorff transform of a function introduced by Rogosinski [5]. Let $\chi(t)$ be a

given function of bounded variation in [0, 1]. Given any function $\phi(t)$ which is measurable and bounded in any finite interval, let

$$\psi(t) = \int_0^1 \phi(tu)\, d\chi(u) = \int_0^t \phi(u)\, d\chi\left(\frac{u}{t}\right). \tag{7}$$

If $\psi(t) \to s$ as $t \to \infty$, we say that the function $\phi(t)$ is summable (H, χ) to s. There is clearly no loss of generality in taking $\chi(0) = 0$; assuming this, (H, χ) is regular (i.e., $\phi(t) \to s$ as $t \to \infty$ implies that $\psi(t) \to s$ as $t \to \infty$) if and only if [1] $\chi(0+) = 0$, $\chi(1) = 1$.

Associated with any Hausdorff transformation (H, χ) there is a Mellin transform $T(z)$, defined for $\boldsymbol{R}z > 0$ by

$$T(z) = \int_0^1 t^z\, d\chi(t). \tag{8}$$

Conversely, given any function $T(z)$ defined for $\boldsymbol{R}z > 0$, we follow Rogosinski in describing it as a Mellin transform if it can be expressed in the form (8).

LEMMA 1. *Let (H, χ_1), (H, χ_2) be two regular Hausdorff transformations, the corresponding Mellin transforms being $T_1(z)$, $T_2(z)$. Soppose that $T_2(z)/T_1(z)$ is also a Mellin transform. Then if $\phi(t)$ is summable (H, χ_1) to s, it is also summable (H, χ_2) to s.*

The result that, under the hypotheses of the lemma, $\varphi(t)$ is summable (H, χ_2) to *some* limit is given by [5], Theorem 2. The result that this limit is s is not included in the explicit statement of that theorem; however, in view of the conditions for regularity already stated, it follows from the proof of that theorem with the aid of equations (4), (5) of § 1.6 of [5].

LEMMA 2. *Let*

$$T(\alpha, \lambda, \beta, \mu; z) = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\beta+1)}{\Gamma(\mu+1)} \frac{\Gamma(z+\alpha+1)}{\Gamma(z+\lambda+1)} \frac{\Gamma(z+\mu+1)}{\Gamma(z+\beta+1)}.$$

If $\alpha > -1$, $\lambda > -1$, $\beta > -1$, $\mu > -1$, $\beta - \mu \geqslant \alpha - \lambda$, then $T(\alpha, \lambda, \beta, \mu; z)$ (as a function of z) is a Mellin transform.

Write

$$\tau(\gamma; z) = \frac{\Gamma(z+\gamma+1)}{\Gamma(\gamma+1)\, \Gamma(z+1)\, (z+1)^\gamma}.$$

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[1] [5], Theorem 1. It is to be noted that Rogosinski uses the term " regular " in a wider sense.

It is known [1] that, if $\gamma > -1$, then $\tau(\gamma; z)$ and its reciprocal are both Mellin transforms. It is also known that, for $\delta \geqslant 0$, $(z+1)^{-\delta}$ is a Mellin transform.

But

$$T(\alpha, \lambda, \beta, \mu; z) = \frac{\tau(\alpha; z)\, \tau(\mu; z)}{\tau(\lambda; z)\, \tau(\beta; z)} (z+1)^{\alpha+\mu-\lambda-\beta}.$$

Since the product of a finite number of Mellin transforms is a Mellin transform, the lemma follows.

3. It is clear that if, for a given $\alpha > -1$, $\lambda > -1$, (6) converges for all $t > 0$, then the same will hold for any other $\alpha > -1$, $\lambda > -1$. Throughout the rest of the paper, this will be assumed to be the case.

Lemma 3. *If* $\alpha > -1$, $\lambda > -1$, *then, for* $t > 0$, $\frac{d}{dt}\{t^{\alpha+1} \phi_\lambda(\alpha+1; t)\} = (\alpha+1)\, t^\alpha \phi_\lambda(\alpha; t)$.

We have (the formal manipulations being justified by absolute convergence),

$$\frac{d}{dt}\{t^{\alpha+1} \phi_\lambda(\alpha+1; t)\} = \frac{d}{dt} \sum_{n=0}^{\infty} \binom{n+\lambda}{n} s_n^{\alpha+1} \frac{t^{n+\alpha+1}}{(1+t)^{n+\lambda+1}} =$$

$$= \sum_{n=0}^{\infty} \binom{n+\lambda}{n} s_n^{\alpha+1} \left\{ (n+\alpha+1) \frac{t^{n+\alpha}}{(1+t)^{n+\lambda+1}} - (n+\lambda+1) \frac{t^{n+\alpha+1}}{(1+t)^{n+\lambda+2}} \right\} =$$

$$= \sum_{n=0}^{\infty} \frac{t^{n+\alpha}}{(1+t)^{n+\lambda+1}} \binom{n+\lambda}{n} \left[(n+\alpha+1)\, s_n^{\alpha+1} - n\, s_{n-1}^{\alpha+1} \right].$$

Since the expression in square brackets is equal to $(\alpha+1)\, s_n^\alpha$, the lemma follows.

As an immediate corollary, we have

$$\phi_\lambda(\alpha+1; t) = (\alpha+1)\, t^{-\alpha-1} \int_0^t u^\alpha \phi_\lambda(\alpha; u)\, du. \qquad (9)$$

It may be remarked that (9) is a special case of the more general result that, for $\beta > \alpha > -1$, $\lambda > -1$,

$$\phi_\lambda(\beta; t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\, \Gamma(\beta-\alpha)} t^{-\beta} \int_0^t (t-u)^{\beta-\alpha-1} u^\alpha \phi_\lambda(\alpha; u)\, du. \qquad (10)$$

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1) This is given, for example, by the proof of [2], Theorem 211.

This reduces to (3) when $\lambda = 0$. However, (10) will not be needed for the proof of the main theorem, so I omit its proof.

4. We now come to the proof of the theorem. In view of the definition of $\phi_\lambda(\alpha; t)$, we see on applying (5) with s_n replaced by s_n^α that, for $\alpha > -1$, $\lambda > \mu > -1$,

$$\phi_\mu(\alpha; t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\,\Gamma(\mu+1)}\, t^{-\lambda} \int_0^t (t-u)^{\lambda-\mu-1}\, u^\mu\, \phi_\lambda(\alpha; u)\, du\,. \quad (11)$$

Consider in particular the special case in which $\lambda = \alpha$. It is well known that

$$\phi_\alpha(\alpha; u) = \phi_0(0; u) = \phi(u)\,;$$

thus, changing the notation by writing λ in place of μ, we find that, for $\alpha > \lambda > -1$

$$\phi_\lambda(\alpha; t) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\lambda)\,\Gamma(\lambda+1)}\, t^{-\alpha} \int_0^t (t-u)^{\alpha-\lambda-1}\, u^\lambda\, \phi(u)\, du. \quad (12)$$

Thus, for $\alpha > \lambda > -1$, $\phi_\lambda(\alpha; t)$ is obtained from $\phi(t)$ by the (H, χ) transformation with

$$\chi(t) = \int_0^t \chi^1(u)\, du\,;$$

$$\chi^1(u) = \chi^1_\lambda(\alpha; u) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\lambda)\,\Gamma(\lambda+1)}\, u^\lambda (1-u)^{\alpha-\lambda-1}\,.$$

The corresponding Mellin transform is

$$T(z) = T_\lambda(\alpha; z) = \frac{\Gamma(\alpha+1)\,\Gamma(z+\lambda+1)}{\Gamma(\lambda+1)\,\Gamma(z+\alpha+1)}\,. \quad (13)$$

Thus, with the notation of Lemma 2,

$$\frac{T_\mu(\beta; z)}{T_\lambda(\alpha; z)} = T(\alpha, \lambda, \beta, \mu; z)\,.$$

By Lemma 2, this is a Mellin transform whenever the appropriate inequalities are satisfied; and the case of the theorem in which $\alpha > \lambda$ therefore follows at once from Lemma 1.

If $\alpha \leqslant \lambda$, however, (12) is no longer valid, and this case of the theorem therefore requires further consideration. We suppose from now on that the inequalities imposed in the theorem are satisfied. Thus, by Lemma 2, $T(\alpha, \lambda, \beta, \mu; z)$ is a Mellin transform, so that we can write

$$T(\alpha, \lambda, \beta, \mu; z) = \int_0^1 t^z \, d\chi(\alpha, \lambda, \beta, \mu; t), \tag{14}$$

say. If, further, $\alpha > \lambda$, the proof of Lemma 1 then shows that

$$\phi_\mu(\beta; t) = \int_0^1 \phi_\lambda(\alpha; tu) \, d\chi(\alpha, \lambda, \beta, \mu; u). \tag{15}$$

We will show that, if for given $\alpha, \lambda, \beta, \mu$, (15) holds with α, β replaced by $\alpha+1$, $\beta+1$, then it holds as it stands. By successive applications of this result, it will then follow that if (15) holds with α, β replaced by $\alpha+r$, $\beta+r$ (r a positive integer), then it holds as it stands; and, since we can choose $\alpha+r>\lambda$, this will give the theorem.

In order to prove the result stated, we write, for the sake of brevity, $\chi(t)$ in place of $\chi(\alpha+1, \lambda, \beta+1, \mu; t)$. We obtain, with the aid of Lemma 3,

$$(\beta+1) t^\beta \phi_\mu(\beta; t) = \frac{d}{dt}\{t^{\beta+1} \phi_\mu(\beta+1; t)\} =$$

$$= \frac{d}{dt}\{t^{\beta+1} \int_0^1 \phi_\lambda(\alpha+1; tu) \, d\chi(u)\} =$$

$$= (\alpha+1) t^\beta \int_0^1 \phi_\lambda(\alpha; tu) \, d\chi(u) + (\beta-\alpha) t^\beta \int_0^1 \phi_\lambda(\alpha+1; tu) \, d\chi(u) =$$

$$= (\alpha+1) t^\beta \int_0^1 \phi_\lambda(\alpha; tu) \, d\chi(u) +$$

$$+ (\beta-\alpha)(\alpha+1) t^\beta \int_0^1 u^\alpha \phi_\lambda(\alpha; tu) \, du \int_u^1 v^{-\alpha-1} \, d\chi(v).$$

Thus

$$\phi_\mu(\beta; t) = \int_0^1 \phi_\lambda(\alpha; tu) \, d\psi(u), \tag{16}$$

where

$$\psi(u) = \frac{(\alpha+1)}{(\beta+1)} \left\{ \chi(u) + (\beta-\alpha) \int_0^u w^\alpha \, dw \int_w^1 v^{-\alpha-1} \, d\chi(v) \right\}. \tag{17}$$

Hence, for $Rz > 0$,

$$\int_0^1 t^z \, d\psi(t) = \frac{(\alpha+1)}{(\beta+1)} \left\{ \int_0^1 t^z \, d\chi(t) + (\beta-\alpha) \int_0^1 t^{z+\alpha} \, dt \int_t^1 v^{-\alpha-1} \, d\chi(v) \right\} =$$

$$= \frac{(\alpha+1)}{(\beta+1)} \left\{ \int_0^1 t^z \, d\chi(t) + (\beta-\alpha) \int_0^1 v^{-\alpha-1} \, d\chi(v) \int_0^v t^{z+\alpha} \, dt \right\} =$$

$$= \frac{(\alpha+1)}{(\beta+1)} \, T(\alpha+1, \lambda, \beta+1, \mu; z) \left\{ 1 + \frac{\beta-\alpha}{z+\alpha+1} \right\}, \qquad (18)$$

by the result obtained by replacing α, β by $\alpha+1$, $\beta+1$ in (14). It now follows at once from the definition of $T(\alpha, \lambda, \beta, \mu; z)$ that

$$\int_0^1 t^z \, d\psi(t) = T(\alpha, \lambda, \beta, \mu; z). \qquad (19)$$

We may suppose $\psi(t)$ normalised by taking

$$\psi(0) = 0; \qquad \psi(t) = \tfrac{1}{2}(\psi(t+) + \psi(t-)) \qquad (0 < t < 1).$$

If $\chi(\alpha, \lambda, \beta, \mu; t)$ is similarly normalised, it follows from (14) and (19) with the aid of the uniqueness theorem for Mellin transforms that

$$\psi(t) = \chi(\alpha, \lambda, \beta, \mu; t).$$

The proof of the theorem is thus completed.

5. It is easily seen that, whenever the transformation (7) is regular, it is also absolutely regular; that is, it transforms any absolutely convergent function (that is to say, a function of bounded variation in $(0, \infty)$) into an absolutely convergent function. The proof of the theorem therefore shows that the result remains true if we replace summability by absolute summability throughout.

REFERENCES

[1] Borwein, D., On a scale of Abel-type summability methods. *Proc. Camb. Phil. Soc.*, 53 (1957), 318-322.

[2] Hardy, G. H. *Divergent series* (Oxford, 1949).

[3] Kogbetliantz, E., Sommation des séries et intégrales divergentes par les moyennes arithmétiques et typiques. *Mém. des Sci. Math.*, fasc. 51 (1931).

[4] Lord, R. D., On some relations between the Abel, Borel and Cesàro methods of summation. *Proc. London Math. Soc* (2), 38 (1935), 241-256.

[5] Rogosinski, W. W., On Hausdorff's methods of summability. II. *Proc. Camb. Phil. Soc.*, 38 (1942), 344-363.

Department of Pure Mathematics
The University
Birmingham 15

(Reçu le 15 Août 1968)

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