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To the memory of J. Karamata

1. Let f be a 2π-periodic and L-integrable function and $(\sigma_n[f])$ the sequence of Fejér-Lebesgue sums of the Fourier series of f, defined by

$$\sigma_n[f](x) = \frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} f(t) \left(\frac{\sin\frac{1}{2}(n+1)(x-t)}{\sin\frac{1}{2}(x-t)} \right)^2 dt .$$

The classical result of Fejér [1] states that if f is a 2π-periodic, bounded and R-integrable function, continuous at the point x, then

$$\lim_{n\to\infty} \sigma_n[f](x) = f(x) \tag{1.1}$$

and the convergence is uniform on any closed interval where f is continuous. In particular, if f is a 2π-periodic and continuous function, then

$$\|\sigma_n[f]-f\| = \sup_{-\infty<x<\infty} |\sigma_n[f](x)-f(x)| \to 0 \quad (n\to\infty). \tag{1.2}$$

Fejér's result has been generalized by Lebesgue [2] who has proved that (1.1) holds whenever f is a 2π-periodic and L-integrable function and

$$\int_0^h |f(x+u) - f(x)|\, du = o(h) \quad (h\to 0). \tag{1.3}$$

At the same time, Lebesgue has shown that this condition is satisfied almost everywhere. In particular, (1.3) holds at each point x where f is continuous.

For 2π-periodic and continuous functions more precise versions of Fejér's result (1.2) usually give estimate of the rate of convergence of the sequence $(\sigma_n[f])$ to f either in terms of the modulus of continuity ω_f which is defined by

$$\omega_f(h) = \sup\{ |f(x') - f(x'')| : |x'-x''| \leqq h \}$$

or in terms of the best approximation $E_n^T(f)$ which is defined by

$$E_n^T(f) = \inf\{\,\|f-t\| : t \in T_n\}$$

where T_n is the set of all trigonometric polynomials of degree $\leqq n$. The following results are known.

If f is a 2π-periodic and continuous function such that $\omega_f(h)\,h^{-\eta}$ is decreasing for some $\eta \in (0, 1)$, then

$$\|\sigma_n[f] - f\| \leqq A\omega_f\left(\frac{1}{n}\right) \tag{1.4}$$

(see [3], Vol. 1, Ch. 3, Th. 3.16). For arbitrary 2π-periodic and continuous f we have the inequality

$$\|\sigma_n[f] - f\| \leqq B\omega_f\left(\frac{\log n}{n}\right) \qquad (n \geqq 2) \tag{1.5}$$

which in a somewhat weaker form was obtained by D. Jackson (see [4], p. 64), and the inequality

$$\|\sigma_n[f] - f\| \leqq \frac{C}{n+1}\sum_{k=0}^{n} \omega_f\left(\frac{1}{k+1}\right) \tag{1.6}$$

which was obtained recently by S. B. Stečkin [5]. Stečkin's inequality follows from a still more precise inequality

$$\|\sigma_n[f] - f\| \leqq \frac{12}{n+1}\sum_{k=0}^{n} E_{k+1}^T(f)$$

and Jackson's theorem which states that $E_n^T(f) \leqq 12\omega_f(1/n)$. Using the well known properties of the modulus of continuity it is easy to see that Stečkin's inequality (1.6) is more precise than (1.5).

The aim of this paper is to obtain the analog of Stečkin's inequality (1.6) for arbitiary 2π-periodic and L-integrable functions and to show that this result cannot be improved for certain classes of functions. As a measure of deviation of $\sigma_n[f](x)$ from $f(x)$ we shall take the function $w[f, x]$ defined by

$$w[f, x](h) = \sup\left\{(2t)^{-1}\int_{-t}^{t} |f(x+u) - f(x)|\,du : 0 < t \leqq h\right\}.$$

Clearly, $w[f, x]$ is a non-decreasing, real valued function and, by (1.3), $w[f, x](h) \to 0$ $(h \to 0)$ almost everywhere. If f is a continuous function we have for $0 < t \leqq h$

$$\int_{-t}^{t} |f(x+u) - f(x)|\,du \leqq \int_{-t}^{t} \omega_f(|u|)\,du \leqq 2t\omega_f(h)$$

and consequently

$$w[f,x](h) \leqq \omega_f(h). \tag{1.7}$$

Our first result can be stated as follows.

THEOREM 1. *Let f be a 2π-periodic and L-integrable function. Then we have for all $n \geqq 0$ the inequality*

$$|\sigma_n[f](x) - f(x)| \leqq \frac{3}{n+1} \sum_{k=0}^{n} w[f,x]\left(\frac{\pi}{k+1}\right). \tag{1.8}$$

For continuous 2π-periodic functions in view of (1.7) we obtain immediately from (1.8) Stečkin's inequality

$$\|\sigma_n[f] - f\| \leqq \frac{3}{n+1} \sum_{k=0}^{n} \omega_f\left(\frac{\pi}{k+1}\right).$$

From Theorem 1 we can obtain similar results valid for all functions in certain classes of 2π-periodic and L-integrable functions. We shall consider here two classes of these functions generated by a non-negative and increasing real valued function Ω defined on $[0, \pi]$ with $\Omega(0) = 0$. The first class $L(\Omega)$ consists of all 2π-periodic and L-integrable functions f such that

$$A_f = \sup_{0<h\leqq\pi} \frac{w[f,x](h)}{\Omega(h)} < \infty.$$

The second, somewhat larger class $L^*(\Omega)$ consists of all 2π-periodic and L-integrable functions f such that

$$B_f = \limsup_{h\to 0+} \frac{w[f,x](h)}{\Omega(h)} < \infty.$$

From (1.8) follows immediately that for any $f \in L(\Omega)$ and $n = 0, 1, 2, \ldots$ we have

$$|\sigma_n[f](x) - f(x)| \leqq \frac{3A_f}{n+1} \sum_{k=0}^{n} \Omega\left(\frac{\pi}{k+1}\right)$$

and so

$$\sup_{n\geqq 0} \left(\frac{|\sigma_n[f](x) - f(x)|}{\frac{1}{n+1} \sum_{k=0}^{n} \Omega\left(\frac{\pi}{k+1}\right)} \right) \leqq 3 \sup_{0<h\leqq\pi} \frac{w[f,x](h)}{\Omega(h)}. \tag{1.9}$$

On the other hand, if $\Sigma\, \Omega(\pi/k+1)$ is a divergent series we have for every $f \in L^*(\Omega)$ the inequality

$$(1.10)\qquad \limsup_{n\to\infty} \frac{|\sigma_n[f](x) - f(x)|}{\dfrac{1}{n+1}\sum_{k=0}^{n} \Omega\left(\dfrac{\pi}{k+1}\right)} \leqq 3 \limsup_{h\to 0+} \frac{w[f,x](h)}{\Omega(h)}.$$

In order to show how (1.10) can be obtained from (1.8), observe that for $k \geqq N_f'$ we have $w[f, x](\pi/k+1) \leqq (B_f+\varepsilon)\,\Omega(\pi/k+1)$. Consequently, from (1.8) follows that for $n \geqq N_f'$

$$|\sigma_n[f](x) - f(x)| \leqq \frac{3}{n+1}\sum_{k=0}^{N_f'} w[f,x]\left(\frac{\pi}{k+1}\right) + $$
$$+ \frac{3(B_f+\varepsilon)}{n+1}\sum_{k=N_f'+1}^{n} \Omega\left(\frac{\pi}{k+1}\right).$$

Since $\Sigma\Omega(\pi/k+1)$ is a divergent series, we can find N_f'' such that for $n > N_f''$

$$\sum_{k=0}^{N_f'} w[f,x]\left(\frac{\pi}{k+1}\right) \leqq \varepsilon \sum_{k=0}^{n} \Omega\left(\frac{\pi}{k+1}\right).$$

Hence, for $n > \max(N_f', N_f'')$ we have

$$|\sigma_n[f](x) - f(x)| \leqq \frac{3(B_f+2\varepsilon)}{n+1}\sum_{k=0}^{n} \Omega\left(\frac{\pi}{k+1}\right)$$

and (1.10) follows.

In order to show that the inequality (1.8) cannot be essentially improved we shall consider the class $L_M(\Omega) \subseteq L(\Omega)$ of 2π-periodic and L-integrable functions f such that

$$A_f = \sup_{0<h\leqq\pi} \frac{w[f,x](h)}{\Omega(h)} \leqq M.$$

We have then the following result.

THEOREM 2. *There exists m $(0 < m < M)$ such that for all $n \geqq 2$ we have the inequalities*

$$(1.11)\qquad \frac{m}{n+1}\sum_{k=2}^{n} \Omega\left(\frac{\pi}{k+1}\right) \leqq \sup_{f\in L_M(\Omega)} |\sigma_n[f](x) - f(x)| \leqq$$
$$\leqq \frac{3M}{n+1}\sum_{k=0}^{n} \Omega\left(\frac{\pi}{k+1}\right).$$

2. *Proof of Theorem 1.* The following proof of Theorem 1 is based on a small modification of the classical proof of the Fejér-Lebesgue's theorem (see [3], Vol. 1, Ch. 3, Th. 3.9).

Let $0 \leqq t \leqq \pi$ and

$$F_x(t) = \int_0^t |f(x+u) + f(x-u) - 2f(x)|\, du .$$

We have then

$$F_x(t) \leqq \int_{-t}^{t} |f(x+u) - f(x)|\, du \leqq 2t w[f, x](t) . \tag{2.1}$$

Since

$$\sigma_n[f](x) - f(x) =$$

$$= \frac{1}{2\pi(n+1)} \int_0^{\pi} (f(x+t) + f(x-t) - 2f(x)) \left(\frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2} t} \right)^2 dt$$

we have

$$| \sigma_n[f](x) - f(x) | \leqq$$

$$\leqq \frac{1}{2\pi(n+1)} \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right) |f(x+t) + f(x-t) - 2f(x)| \left(\frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2} t} \right)^2 dt$$

$$\leqq P_n + Q_n .$$

Since $|\sin nt| \leqq n |\sin t|$ we have

$$P_n \leqq \frac{1}{2\pi}(n+1) \int_0^{\pi/n+1} |f(x+t) + f(x-t) - 2f(x)|\, dt = \tag{2.2}$$

$$= \frac{1}{2\pi}(n+1) F_x\left(\frac{\pi}{n+1} \right) .$$

Next, since $\sin \frac{1}{2} t \geqq t/\pi$ for $t \in [0, \pi]$, we have

$$Q_n \leqq \frac{\pi}{2(n+1)} \int_{\pi/n+1}^{\pi} |f(x+t) + f(x-t) - 2f(x)| \frac{dt}{t^2} .$$

Using partial integration we find that

$$(2.3) \qquad Q_n \leqq \frac{1}{2\pi(n+1)} F_x(\pi) - \frac{1}{2\pi}(n+1) F_x\left(\frac{\pi}{n+1}\right) + \\ + \frac{\pi}{n+1} \int_{\pi/n+1}^{\pi} F_x(t) \frac{dt}{t^3}.$$

Adding (2.2) and (2.3) and using (2.1) we find that

$$(2.4) \qquad | \sigma_n[f](x) - f(x) | \leqq \frac{1}{n+1} w[f,x](\pi) + \\ + \frac{2\pi}{n+1} \int_{\pi/n+1}^{\pi} w[f,x](t) \frac{dt}{t^2}.$$

Since $w[f, x]$ is non-negative and non-decreasing we have

$$w[f,x](\pi) \leqq \sum_{k=0}^{n} w[f,x](\pi/k+1)$$

and

$$\int_{\pi/n+1}^{\pi} w[f,x](t) \frac{dt}{t^2} = \frac{1}{\pi} \int_{1}^{n+1} w[f,x]\left(\frac{\pi}{t}\right) dt \leqq \frac{1}{\pi} \sum_{k=0}^{n} w[f,x]\left(\frac{\pi}{k+1}\right)$$

and (1.8) follows from the last two inequalities and (2.4).

Proof of Theorem 2. The right hand side inequality (1.11) follows immediately from (1.9). To prove the left hand side inequality (1.11) let $g_x(t) = M\,\Omega(|t-x|)$ if $|t-x| \leqq \pi$ and $g_x(t+2\pi) = g_x(t)$. Then g_x is clearly a 2π-periodic and L-integrable function. Since for $0 < t \leqq h \leqq \pi$ we have

$$\int_{-t}^{t} | g_x(x+u) - g_x(x) | \, du = M \int_{-t}^{t} \Omega(|u|) \, du \leqq 2t\, M\Omega(h),$$

it follows that $w[g_x, x](h) \leqq M\,\Omega(h)$ and so $g_x \in L_M(\Omega)$.

Next,

$$(2.5) \qquad \sup_{f \in L_M(\Omega)} | \sigma_n[f](x) - f(x) | \geqq |\sigma_n[g_x](x) - g_x(x)| = \sigma_n[g_x](x).$$

We have first

$$\sigma_n[g_x](x) = \frac{1}{2\pi(n+1)} \int_0^{\pi} (g_x(x+t) + g_x(x-t)) \left(\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right)^2 dt$$

$$= \frac{M}{\pi(n+1)} \int_0^{\pi} \Omega(t) \left(\frac{sin \frac{1}{2}(n+1)t}{sin \frac{1}{2} t} \right)^2 dt$$

$$\geqq \frac{4M}{\pi(n+1)} \int_{\frac{\pi}{n+1}}^{\pi} \Omega(t) \left(\frac{sin \frac{1}{2}(n+1)t}{sin \frac{1}{2} t} \right)^2 dt .$$

Since $| \sin x | \leqq | x |$ it follows that

$$\sigma_n [g_x](x) \geqq \frac{4M}{\pi(n+1)} \int_{\pi/n+1}^{\pi} \Omega(t) (sin \tfrac{1}{2}(n+1)t)^2 \frac{dt}{t^2} = A_n . \tag{2.6}$$

Since Ω is non-decreasing, we have

$$A_n = \frac{4M}{\pi(n+1)} \sum_{k=1}^{n} \int_{\frac{k\pi}{n+1}}^{\frac{(k+1)\pi}{n+1}} \Omega(t) (sin \tfrac{1}{2}(n+1)t)^2 \frac{dt}{t^2}$$

$$\geqq \frac{4M}{\pi(n+1)} \sum_{k=1}^{n} \Omega\left(\frac{k\pi}{n+1}\right) \left(\frac{n+1}{k+1}\right)^2 \frac{1}{\pi^2} \int_{\frac{k\pi}{n+1}}^{\frac{(k+1)\pi}{n+1}} (sin \tfrac{1}{2}(n+1)t)^2 dt .$$

Since

$$\int_{\frac{k\pi}{n+1}}^{\frac{(k+1)\pi}{n+1}} (sin \tfrac{1}{2}(n+1)t)^2 dt = \frac{\pi}{2(n+1)} ,$$

we find that for $n \geqq 2$

$$A_n \geqq 2M\pi^{-2} \sum_{k=1}^{n} \frac{1}{(k+1)^2} \Omega\left(\frac{k\pi}{n+1}\right) \geqq 2M\pi^{-2} \sum_{k=2}^{n} \frac{1}{(k+1)^2} \Omega\left(\frac{k\pi}{n+1}\right) .$$

Next, for $2 \leqq k \leqq n$

$$\int_{\frac{(k-1)\pi}{n+1}}^{\frac{k\pi}{n+1}} \Omega(t) \frac{dt}{t^2} \leqq \frac{n+1}{\pi} \cdot \frac{1}{k(k-1)} \Omega\left(\frac{k\pi}{n+1}\right)$$

and so

$$\frac{1}{(k+1)^2}\,\Omega\left(\frac{k\pi}{n+1}\right) \geqq \frac{\pi}{n+1}\,\frac{k(k-1)}{(k+1)^2}\int_{\frac{(k-1)\pi}{n+1}}^{\frac{k\pi}{n+1}} \Omega(t)\frac{dt}{t^2} \geqq \frac{\pi}{5(n+1)}\int_{\frac{(k-1)\pi}{n+1}}^{\frac{k\pi}{n+1}} \Omega(t)\frac{dt}{t^2}.$$

Hence, for $n \geqq 2$,

$$A_n \geqq \frac{2M}{5\pi(n+1)}\sum_{k=2}^{n}\int_{\frac{(k-1)\pi}{n+1}}^{\frac{k\pi}{n+1}} \Omega(t)\frac{dt}{t^2} \geqq \frac{2M}{5\pi(n+1)}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{2}} \Omega(t)\frac{dt}{t^2}.$$

Since Ω is non-decreasing, we have

$$\int_{\pi/n+1}^{\pi/2} \Omega(t)\frac{dt}{t^2} = \frac{1}{\pi}\int_{2}^{n+1} \Omega\left(\frac{\pi}{t}\right)dt \geqq \frac{1}{\pi}\sum_{k=2}^{n}\Omega\left(\frac{\pi}{k+1}\right).$$

Hence

$$A_n \geqq \frac{2M}{5\pi^2(n+1)}\sum_{k=2}^{n}\Omega\left(\frac{\pi}{k+1}\right) \tag{2.7}$$

and the left hand side inequality (1.11) follows from (2.5), (2.6) and (2.7).

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