SOME CONVERSE THEOREMS ON THE ABSCISSAE OF SUMMABILITY OF GENERAL DIRICHLET SERIES

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To the memory of J. Karamata

INTRODUCTION

Chandrasekharan and Minakshisundaram have generalized ([6], p. 21, Theorem 1.82) a fundamental theorem which asserts the convergence of a series when the series is (i) summable by a Riesz mean of general type λ and some positive order, (ii) subject to an appropriate Tauberian condition in two-sided Schmidt form. Basing themselves on their generalization, they have extended at one stroke ([6], pp. 86, 88, Theorems 3.71, 3.72), certain converse theorems on the abscissae of summability of general Dirichlet series, due in the first instance to Ananda-Rau ([2], Theorems 7, 8, 9) with Tauberian conditions on individual coefficients of the series, and due subsequently to Ganapathy Iyer ([7], Theorems 7, 8, 10) with Tauberian conditions formally including those of Ananda-Rau. Now the fundamental theorem generalized by Chandrasekharan and Minakshisundaram contains, besides the two-sided Schmidt hypothesis taken into account by them, an alternative one-sided hypothesis. And this theorem in its entirety, with both alternative hypotheses, has a natural generalization in Theorem A (§ 1) of which it is, in fact, the special case $a = b = 0$. In the present context the significance of Theorem A lies in its being a basis, not only for the extensions of Ananda-Rau's and Ganapathy Iyer's theorems given by Chandrasekharan and Minakshisundaram, but also for some further extensions of the same type (§§ 2, 3, 4).

It is relevant to mention here that the earliest version of Theorem A is due to Karamata ([8], § 1.1) and concerned with the Cesàro first-order mean of a series or sequence in place of a Riesz mean of general type λ and some positive order. Two later versions, also due to Karamata and found in a paper by him dated November 1939 ([9], Théorèmes 1a), 3f)), are concerned with an integral mean including as a special case a Riesz mean of general type λ and some *positive integer* order. These later versions are proved by him by using a difference formula applicable to such an integral mean ([9], Lemma 2); and each of them has a hypothesis which is an extension of the one-sided or two-sided Schmidt condition of slow growth of a function. Theorem A is a reformulation of Karamata's later theorems for any Riesz mean of a sequence, of general type λ and some *positive non-integer order*. In its fundamental case, $a = b = 0$, Theorem A has an analogue for the Abel mean of type λ instead of a Riesz mean of type λ, consisting of a classical theorem ([5], Theorem E) and Bosanquet's addition thereto ([5], Theorem D). Theorem A itself has been proved by me ([12], Theorem VI) by means of certain difference formulae due to Bosanquet ([4], Theorem 1) which extend Karamata's difference formula just mentioned to an integral mean of *non-integer* order. Bosanquet first proved his extended difference formulae in 1943, independently of Karamata. But, as a matter of fact, he had used them much earlier in 1931 in a form equivalent to Karamata's ([3], Lemma 5). To complete the references in relation to Bosanquet's difference formulae, mention may be made of certain other difference formulae independently evolved by Minakshisundaram and myself ([10], formulae (2.32), (2.38)) which are serviceable for much the same purposes as Bosanquet's formulae.

This paper deals specifically with general Dirichlet series of type l as distinguished from those of type λ. However, as far as Riesz typical means alone are concerned, there is no distinction between means of the two types, and so (for convenience) the Riesz means of this paper are taken to be of type l or (more explicitly) of type l_n, where l or l_n ($n = 1, 2, \ldots$) is a divergent sequence strictly increasing and positive.

§ 1. Notation and auxiliary results

Let $a_1+a_2+\ldots$ be a real series and l a sequence $\{l_n\}$ such that

$$1 \leqslant l_1 < l_2 < \ldots, \quad l_n \to \infty .$$

Then, as usual, we define the Riesz mean of Σa_n of type l or l_n and order $r>0$ by

$$\int_0^x \left(1 - \frac{t}{x}\right)^r dA_l(t) = \frac{r}{x^r} \int_0^x (x-t)^{r-1} A_l(t)\, dt \equiv \frac{A_l^r(x)}{x^r},$$

where $A_l^r(x)$ is the usual Riesz sum of Σa_n of type l or l_n and order r,

$$A_l(t) = a_1 + a_2 + \dots + a_n \quad \text{for} \quad l_n \leqslant t < l_{n+1} \ (n \geqslant 1),$$

$$A_l(t) = 0 \quad \text{for} \quad t < l_1 .$$

Again, as usual, we define $A_l^0(t) = A_l(t)$ and define as follows summability of Σa_n to sum S by the Riesz mean of type l_n and order $r \geqslant 0$, briefly called summability (R, l_n, r) of Σa_n to S:

$$\frac{A_l^r(x)}{x^r} \to S, \text{ or } A_l^r(x) - Sx^r = o(x^r), \ x \to \infty .$$

In using this definition we may suppose (without loss of generality) that $S=0$ since this merely means our considering $\Sigma a_n - S$ instead of Σa_n. Furthermore, when considering any other series $b_1+b_2+\dots$, it is convenient to denote by $B_l^r(x)$, $r \geqslant 0$, the Riesz sum for that series, defined exactly as $A_l^r(x)$ for Σa_n.

In the usual notation again, the general Dirichlet series of type l or l_n, with coefficients $\{a_n\}$, is

$$\sum_1^\infty \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau .$$

Corresponding to the summability (R, l_n, r), $r \geqslant 0$, of this series, to sum-function $f(s)$, we have the abscissa of summability σ_r $(-\infty < \sigma_r < \infty)$ characterized by the property that the series is summable (R, l_n, r), or not summable (R, l_n, r), according as $\sigma > \sigma_r$ or $\sigma < \sigma_r$.

In the above notation, we may state as under the lemmas and auxiliary theorems used in this paper, denoting Riesz sums of order $r \geqslant 0$, of Σa_n and Σb_n respectively, by $A^r(x)$ and $B^r(x)$, with omission of the suffix l indicative of the type which remains the same throughout.

LEMMA 1 ([1], Theorem 6; [2], Theorem 1). *Let* $\Sigma b_n \equiv \Sigma a_n l_n^\gamma$, *where* $\gamma > 0$ *is a constant. If* $A^r(x) = o(x^\beta)$, $x \to \infty$, *where* $\beta \geqslant r \geqslant 0$, *then* $B^r(x) = o(x^{\beta+\gamma})$.

LEMMA 2 ([1], Theorem 9; [2], Theorem 3). *If* $A^k(x) = o(x^{k+\beta})$, $x \to \infty$, *where* $k \geqslant 0$, $\beta > 0$, *then* $\Sigma b_n \equiv \Sigma a_n l_n^{-\beta}$ *is either summable* (R, l_n, k) *or never summable* (R, l_n, r) *for any* r *however large.*

LEMMA 3 ([1], Theorem 4; [2], Theorem II). *If*

$$A^r(x) = o\{W(x)\} \ (r > 0), \ A(x) = O\{V(x)\}, \ x \to \infty ,$$

where $W(x)$, $V(x)$ *are positive monotonic increasing functions of* $x>0$, *then, for* $0<k<r$, $A^k(x) = o(V^{1-k/r}\, W^{k/r})$ *where* $V = V(x)$, $W = W(x), x\to\infty$.

Lemma 3 is proved, in the papers referred to ([1], [2]), for any integrable function $\phi(x)$ instead of $A(x) \equiv \sum_{l_n \leq x} a_n$.

THEOREM A ([12], Theorem VI). *If*

$$A^r(x) = o(x^{r+b}),\ x\to\infty,\ \text{where}\ r>0,\ r+b\geqslant 0, \tag{1.1}$$

and if, with

$$\theta(x) \equiv x^{1-(a-b)/r},\ a\geqslant b,$$

we have

EITHER (a)

$$\overline{\lim_{n\to\infty}} \max_{l_n\leq l_m< l_n+\varepsilon\theta(l_n)} \frac{a_n+a_{n+1}+\dots+a_m}{l_n^a} = o_R(1),\ \varepsilon\to 0,$$

OR (b)

$$\overline{\lim_{n\to\infty}} \max_{l_n\leq l_m< l_n+\varepsilon\theta(l_n)} \frac{|a_{n+1}+a_{n+2}+\dots+a_m|}{l_n^a} = o(1),\ \varepsilon\to 0, \text{[1]} \tag{1.2}$$

then

$$A(l_n) = o(l_n^a),\ n\to\infty.$$

THEOREM B (Riesz; see e.g. [6], p. 81, Theorem 3.66). *Suppose that the Dirichlet series*

$$\sum_1^\infty \frac{a_n}{l_n^s},\quad s=\sigma+i\tau,$$

is summable (R, l_n, q) *for some* $q\geqslant 0$ *when* $\sigma>d$. [2] *Suppose also that the sum-function* $f(s)$ *thus defined is regular for* $\sigma>\eta$ *where* $\eta<d$, *and*

$$f(s) = O(|\tau|^r),\ r\geqslant 0,\quad \text{uniformly for}\quad \sigma\geqslant\eta+\varepsilon>\eta.$$

Then the Dirichlet series is summable (R, l_n, r'), $r'>r$, *for* $\sigma>\eta$.

1) $a_{n+1}+a_{n+2}+\dots+a_m$ is to be interpreted as 0 when $n=m$ or $l_n=l_m$.
2) This is no restriction since otherwise $\sigma_q=\infty$ for all q.

§ 2. A BASIC THEOREM

The theorem which follows supplies a basis for all the other theorems of this paper, whether by itself or not.

THEOREM I. (A) *For the Dirichlet series*

$$\sum_{1}^{\infty} \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau,$$

suppose that $\sigma_r < \rho$ *for some* $r > 0$. *Suppose also that there is a* γ *and an associated* $\theta(x)$ *such that*

$$\sigma_r < \gamma < \rho, \quad \theta(x) \equiv x^{1-(\rho-\gamma)/r}, \tag{2.1}$$

with

EITHER (a)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^\rho} = o_R(1), \ \varepsilon \to 0,$$

OR (b)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < \varepsilon\theta(l_n)} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \ \varepsilon \to 0. \tag{2.2}$$

Then

$$\sigma_k \leqslant \frac{(r-k)\rho + k\sigma_r}{r} \quad (0 \leqslant k < r). \tag{2.3}$$

(B) *If* $\sigma_r \geqslant \rho$, *instead of* $\sigma_r < \rho$ *as in* (A), *and* ρ *is such that*

EITHER (a)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < \varepsilon l_n} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^\rho} = o_R(1), \ \varepsilon \to 0,$$

OR (b)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \ \varepsilon \to 0, \tag{2.4}$$

then

$$\sigma_k = \sigma_r \ (0 \leqslant k < r). \tag{2.5}$$

Proof. (A) The proof is given below only for the case in which $\{a_n\}$ satisfies the hypothesis in alternative (2.2) (a), the remaining case of alternative (2.2) (b) being exactly similar.

By the definition of σ_r and that of γ in (2.1), $\Sigma a_n l_n^{-\gamma}$ is summable (R, l_n, r) to sum S (say), and so

$$b_1+b_2+\dots = (a_1 l_1^{-\gamma}-S) + a_2 l_2^{-\gamma} + \dots \text{ is summable (R, } l_n, r) \text{ to } 0,$$

i.e.,

$$B^r(x) = o(x^r), \quad x \to \infty, \tag{2.6}$$

while it is easy to prove that

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \frac{b_n + b_{n+1} + \dots + b_m}{l_n^{\rho-\gamma}} = o_R(1), \quad \varepsilon \to 0, \tag{2.7}$$

distinguishing between the case $\gamma \geqslant 0$ and $\gamma < 0$. If $\gamma \geqslant 0$, then

$$b_n + b_{n+1} + \dots + b_m = a_n l_n^{-\gamma} + a_{n+1} l_{n+1}^{-\gamma} + \dots + a_m l_m^{-\gamma} \ (n > 1)$$

$$\leqslant l_n^{-\gamma} \max_{n \leqq \nu \leqq m} (a_n + a_{n+1} + \dots + a_\nu),$$

from which and (2.2) (a) we have (2.7) as an immediate consequence. On the other hand, if $\gamma < 0$, then

$$b_n + b_{n+1} + \dots + b_m \leqslant l_m^{-\gamma} \max_{n \leqq \nu \leqq m} (a_\nu + a_{\nu+1} + \dots + a_m), \tag{2.8}$$

where

$$l_n \leqslant l_\nu \leqslant l_m < l_n + \varepsilon\,\theta(l_n) < (1+\varepsilon)\, l_n,$$

$$\frac{\theta(l_n)}{\theta(l_\nu)} = \left(\frac{l_n}{l_\nu}\right)^{(r-\rho+\gamma)/r} < (1+\varepsilon)^{|r-\rho+\gamma|/r} = K \text{ (say)}.$$

Hence (2.8) gives us

$$\max_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \frac{b_n + b_{n+1} + \dots + b_m}{l_n^{\rho-\gamma}}$$

$$\leqslant \frac{l_m^{-\gamma}}{l_n^{\rho-\gamma}} \max_{l_\nu \leqq l_m < l_n + \varepsilon\theta(l_n)} (a_\nu + a_{\nu+1} + \dots + a_m)$$

$$\leqslant \left(\frac{l_m}{l_\nu}\right)^{-\gamma} \left(\frac{l_\nu}{l_n}\right)^{\rho-\gamma} \max_{l_\nu \leqq l_m < l_\nu + \varepsilon K\theta(l_\nu)} \frac{a_\nu + a_{\nu+1} + \dots + a_m}{l_\nu^{\rho}}. \tag{2.9}$$

In (2.9),

$$\left(\frac{l_m}{l_\nu}\right)^{-\gamma} \leqslant 1 , \left(\frac{l_\nu}{l_n}\right)^{\rho-\gamma} < (1+\varepsilon)^{\rho-\gamma}$$

since $\gamma<0$, $\rho-\gamma>0$. Hence (2.9) in conjunction with hypothesis (2.2) (a) leads to (2.7) again.

After this we appeal to Theorem A with (1.1) and (1.2) (a) replaced by (2.6) and (2.7) respectively, to obtain

$$B(x) \equiv B(l_n) = o(l_n^{\rho-\gamma}) = o(x^{\rho-\gamma}) , \quad l_{n+1} > x \geqslant l_n \to \infty .$$

From the last step and (2.6), we get, by using Lemma 3,

$$B^k(x) = o\left\{x^{\left(1-\frac{k}{r}\right)(\rho-\gamma)+\frac{k}{r}r}\right\} \text{ for } 0 \leqslant k < r$$

$$= o(x^{k+\beta}) \text{ where } \beta = \left(1-\frac{k}{r}\right)(\rho-\gamma) > 0 .$$

Hence $\Sigma a_n l_n^{-\gamma-\beta}$, being summable (R, l_n, r), is also summable (R, l_n, k) by Lemma 2, i.e., $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) for

$$\sigma \geqslant \gamma + \beta = \gamma + \left(1-\frac{k}{r}\right)(\rho-\gamma) = \left(1-\frac{k}{r}\right)\rho + \frac{k}{r}\gamma . \tag{2.10}$$

Since $\gamma>\sigma_r$ may be taken as near to σ_r as we please [1]), (2.10) immediately gives us the conclusion (2.3).

In arriving at (2.3) we have tacitly assumed that $\sigma_r > -\infty$. When $\sigma_r = -\infty$, we still reach (2.3) in the sense that $\sigma_k = -\infty$ for $0 \leqslant k < r$, as we may see by taking $\gamma = -G$ (G positive and arbitrarily large) in the preceding argument.

(B) As in (A), we confine ourselves to the hypothesis (2.4) (a), the treatment of (2.4) (b) being precisely similar. Defining as in (A)

$$b_1 + b_2 + \ldots \equiv (a_1 l_1^{-\gamma} - S) + a_2 l_2^{-\gamma} + \ldots \quad (\gamma > \sigma_r) ,$$

we see that (2.6) holds again, while (2.4) (a) implies

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{(b_n + b_{n+1} + \ldots + b_m)}{l_n^{\rho-\gamma}} = o_R(1) , \quad \varepsilon \to 0 ,$$

1) The truth of hypothesis (2.2) for some γ, $\sigma_r < \gamma < \rho$, implies its truth for any γ', $\sigma_r < \gamma' < \gamma$, so that γ may be replaced by γ'.

exactly as (2.2) (a) implies (2.7). Since now $\gamma > \sigma_r \geqslant \rho$, the above condition in its turn implies

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} (b_n + b_{n+1} + \ldots + b_m) = o_R(1), \ \varepsilon \to 0 .$$

By Theorem A with hypothesis (1.2) (a) and $a = b = 0$, it follows that $\Sigma a_n l_n^{-s}$ is convergent for any σ such that $\sigma \geqslant \gamma > \sigma_r$ and therefore $\sigma_0 \leqslant \sigma_r$. But, in any case, $\sigma_0 \geqslant \sigma_k \geqslant \sigma_r$ for $0 \leqslant k < r$ and so we have the conclusion (2.5).

In the preceding argument we have supposed that $\sigma_r < \infty$ since $\sigma_r = \infty$ implies trivially $\sigma_k = \infty$.

§ 3. Applications to theorems of the Schnee-Landau type

Theorem II given next is the simplest of the theorems of the type mentioned above and it is a direct combination of Theorems I, B. Theorems V, VI are generalizations, respectively of Ananda-Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem ([2], Theorem 9; [7], Theorem 10), as given by Chandrasekharan and Minakshisundaram ([6], pp. 88-9, Corollaries 3.73, 3.74). Theorems III, IV are apparently new counterparts of Theorems V, VI, the newness consisting in the replacement of the two-sided Tauberian conditions of the latter pair of theorems by analogous one-sided conditions suitably supplemented.

Theorem II. *Suppose that* (i) *the Dirichlet series,*

$$\sum_1^\infty \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau ,$$

is summable (R, l_n, q) *for some* $q \geqslant 0$ *when* $\sigma > \rho$, (ii) *the sum-function* $f(s)$ *thus defined is regular for* $\sigma > \eta$ *when* $\eta < \rho$, *and satisfies the condition*

$$f(s) = O(|\tau|^r), \ r > 0, \quad \textit{uniformly for} \quad \sigma \geqslant \eta + \varepsilon > \eta,$$

(iii) *the coefficients* a_n *of the Dirichlet series satisfy* ONE *of the two alternatives* (a), (b) *of* (2.2), *but with* $\theta(x) \equiv x^{1-(\rho-\eta)/r}$. *Then the Dirichlet series is summable* (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)\,\rho + k\eta}{r} .$$

Proof. By Theorem B, the Dirichlet series is summable (R, l_n, r'), $r' > r$, for $\sigma > \eta$ and hence $\sigma_{r'} \leqslant \eta < \rho$. Therefore it is evident from the proof of

Theorem I (A) ending with (2.10) that the Dirichlet series is summable (R, l_n, k), $0 \leqslant k < r'$, for

$$\sigma \geqslant \frac{(r'-k)\rho + k\eta}{r'},$$

whence the desired conclusion follows when we let $r' \to r$.

THEOREM III. *In Theorem* II, *let* ρ *be replaced by* $\alpha+1$ *in hypotheses* (i) *and* (ii)*; also let hypothesis* (iii) *be replaced by*

$$a_n = O_R\left[l_n^\alpha (l_n - l_{n-1})\right], \quad l_n - l_{n-1} = O\left(l_n^{\frac{r-\alpha+\eta}{r+1}}\right). \tag{3.1}$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$, $s = \sigma + i\tau$, *is summable* (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)(\alpha+1) + (k+1)\eta}{r+1}. \tag{3.2}$$

Proof. As in the proof of Theorem II, the series $\Sigma a_n l_n^{-s}$ is summable (R, l_n, r'), $r' > r$, for $\sigma > \eta$ where now $\eta < \alpha+1$, so that $\sigma_{r'} \leqslant \eta < \alpha+1$. We begin by choosing γ and correspondingly $\theta(x)$ as follows:

$$\eta < \gamma < \alpha + 1, \quad \theta(x) \equiv x^{(r'-\alpha+\gamma)/(r'+1)}. \tag{3.3}$$

Then, since $r' > r$ and $\gamma > \eta$, we have

$$\frac{r'-\alpha+\gamma}{r'+1} > \frac{r-\alpha+\gamma}{r+1} > \frac{r-\alpha+\eta}{r+1}.$$

And so (3.1) gives us, as $n \to \infty$,

$$a_n = O_R\left[l_n^\alpha l_n^{\frac{r-\alpha+\eta}{r+1}}\right] = o_R\left[l_n^\alpha l_n^{\frac{r'-\alpha+\gamma}{r'+1}}\right] = o_R\left[l_n^\alpha \theta(l_n)\right]. \tag{3.4}$$

Also, if $l_n \leqslant l_m < l_n + \varepsilon\theta(l_n)$, (3.1) again gives us as $n \to \infty$,

$$a_{n+1} + a_{n+2} + \dots + a_m = \begin{cases} O_R\left[l_m^\alpha (l_m - l_n)\right] \text{ if } \alpha \geqslant 0, \\ O_R\left[l_n^\alpha (l_m - l_n)\right] \text{ if } \alpha < 0, \end{cases}$$

so that, whether $\alpha \geqslant 0$ or $\alpha < 0$,

$$a_{n+1} + a_{n+2} + \dots + a_m = O_R\left[l_n^\alpha \varepsilon\, \theta(l_n)\right]. \tag{3.5}$$

In (3.4) and (3.5),

$$l_n^{\alpha}\,\theta(l_n) = l_n^{\rho'} \quad \text{where} \quad \rho' = \alpha + \frac{r' - \alpha + \gamma}{r' + 1}\ (>\gamma)\,.$$

Hence, combining (3.4) and (3.5), we get

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1)\,, \quad \varepsilon \to 0\,. \tag{3.6}$$

(3.6) and the fact, following from Theorem B, that $\Sigma a_n\, l_n^{-s}$ is summable (R, l_n, r'), enables us to use (2.10) in the proof of Theorem I (A) with r, ρ replaced by r', ρ' respectively, so as to infer that $\Sigma a_n\, l_n^{-s}$ is summable (R, l_n, k), $0 \leqslant k < r'$, for

$$\sigma \geqslant \frac{(r'-k)\,\rho' + k\gamma}{r'} = \frac{(r'-k)(\alpha+1) + (k+1)\gamma}{r'+1}\,.$$

This yields (3.2) as required when we let $r' \to r$ and recall that $\gamma\ (>\eta)$ can be taken arbitrarily close to η.

THEOREM IV. *In Theorem* III, (3.1) *alone can be changed to*

$$\left.\begin{aligned} &\sum_{\nu=1}^{n} (a_\nu + |a_\nu|)\, l_\nu^p\, (l_\nu - l_{\nu-1})^{1-p} = O\,(l_n^{p(\alpha+1)+1})^{1}),\ l_n - l_{n-1} = \\ &= O\left[l_n^{\frac{r-\alpha-p^{-1}+\eta}{r+1-p^{-1}}}\right], \quad p > 1\,,\ \alpha+1+p^{-1} \geqslant 0, \end{aligned}\right\} \tag{3.7}$$

with the conclusion changed in consequence to the assertion that $\Sigma a_n\, l_n^{-s}$ *is summable* (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)(\alpha+1) + (k+1-p^{-1})\,\eta}{r+1-p^{-1}}\,. \tag{3.8}$$

Proof. We observe that Theorem III may be viewed as the limiting case $p = \infty$ of Theorem IV.

The proof itself is similar to that of Theorem III with the difference that the choice of γ and $\theta(x)$ in (3.3) is now altered as below:

$$\eta < \gamma < \alpha + 1\,,\ \theta(x) \equiv x^{(r'-\alpha-p^{-1}+\gamma)/(r'+1-p^{-1})}$$

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1) We suppose that $l_0 = 0$.

And furthermore the step corresponding to (3.6) is obtained as follows. Writing $1-1/p = 1/p'$, we get, for $l_n \leqslant l_m < l_n + \varepsilon\theta(l_n)$,

$$a_{n+1} + a_{n+2} + \dots + a_m \leqslant a_{n+1} + |a_{n+1}| + \dots + a_m + |a_m|$$

$$= \sum_{\nu=1}^{m-n} (a_{\nu+n} + |a_{\nu+n}|)\, l_{\nu+n} (l_{\nu+n} - l_{\nu+n-1})^{(1-p)/p} \times$$

$$\times \frac{(l_{\nu+n} - l_{\nu+n-1})^{1/p'}}{l_{\nu+n}}$$

$$\leqslant \left[\sum_{\nu=1}^{m-n} (a_{\nu+n} + |a_{\nu+n}|)^p\, l_{\nu+n}^p (l_{\nu+n} - l_{\nu+n-1})^{1-p} \right]^{1/p} \times$$

$$\times \left[\sum_{\nu=1}^{m-n} \frac{l_{\nu+n} - l_{\nu+n-1}}{l_{\nu+n}^{p'}} \right]^{1/p'}$$

$$= O\left[l_m^{\alpha+1+1/p} \frac{(l_m - l_n)^{1/p'}}{l_{n+1}} \right] \quad (n \to \infty)$$

$$= O\left[l_n^{\alpha+1+1/p} \frac{\{\varepsilon\,\theta(l_n)\}^{1/p'}}{l_n} \right] \tag{3.9}$$

where we have used the hypothesis (3.7) in the passage to the step preceding (3.9). Taking $m = n+1$ in the step preceding (3.9), we get also

$$a_{n+1} = O_R\left[l_n^{\alpha+1+1/p} \frac{(l_{n+1} - l_n)^{1/p'}}{l_{n+1}} \right] \quad (n \to \infty)$$

$$= O_R\left[l_{n+1}^{\alpha+1/p}\, l_{n+1}^{(r-\alpha-p^{-1}+\eta)/(r+1-p^{-1})p'} \right]$$

$$= o_R\left[l_{n+1}^{\alpha+1/p} \{\theta(l_{n+1})\}^{1/p'} \right]. \tag{3.10}$$

From (3.9) and (3.10) with $n+1$ changed to n, we obtain, instead of (3.6) in the proof of Theorem III,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^{\rho'}} = o_R(1), \quad \varepsilon \to 0,$$

where

$$\rho' = \alpha + \frac{1}{p} + \frac{(r' - \alpha - p^{-1} + \gamma)}{(r' + 1 - p^{-1})\, p'}.$$

After this the proof is completed exactly like that of Theorem III subsequent to (3.6).

It may be observed that the assumption $\alpha+1+p^{-1}\geqslant 0$ involves no loss of generality since $\alpha+1+p^{-1}<0$ makes successively $a_n+|a_n|\equiv 0$, $a_n\equiv 0$ and so $\sigma_r = -\infty$ for all $r\geqslant 0$.

THEOREM V. *In Theorem* II, *let hypothesis* (i) *be omitted on account of its being implicit* (*with* $q=0$, $\rho=\alpha+1$) *in hypothesis* (iii) *modified as under. Let hypothesis* (ii) *be retained with* ρ *changed to* $\alpha+1$, *and hypothesis* (iii) *replaced by*

$$a_n = O\left[l_n^{\alpha}(l_n-l_{n-1})\right]. \tag{3.11}$$

Then the conclusion is that $\Sigma a_n l_n^{-s}$ *is summable* (R, l_n, k), $0\leqslant k<r$, *for* σ *satisfying* (3.2).

THEOREM VI. *If, in Theorem* V, (3.11) *alone is changed to*

$$\sum_{\nu=1}^{n} |a_\nu|^p l_\nu^p (l_\nu-l_{\nu-1})^{1-p} = O\left[l_n^{p(\alpha+1)+1}\right],\ p>1,\ \alpha+1+p^{-1}\geqslant 0,$$

the conclusion will become the assertion that $\Sigma a_n l_n^{-s}$ *is summable* (R, l_n, k), $0\leqslant k<r$, *for* σ *satisfying* (3.8).

The proofs of Theorems V, VI are omitted, being obvious simplifications of those of Theorems III, IV, involving the use of Theorem I (A) with hypothesis (2.2) (b) instead of (2.2.) (a) as formerly. Theorems V and VI, as pointed out by Chandrasekharan and Minakshisundaram, yield Ananda Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem when $\alpha\to+0$.

§ 4. FURTHER APPLICATIONS

Theorem I (A) is a base which, combined with Theorem B, produces Theorem II, and in this sense Theorem I (A) may be said to correspond to Theorem II. There are results corresponding to each of Theorems III-VI in the same sense. For instance, Deduction 1 below corresponds to Theorem III and shows how other deductions corresponding to Theorems IV-VI may be formulated. Deductions 2,3 are further examples of results based on Theorem I.

DEDUCTION 1. (A) *In Theorem* I (A), *suppose that* $\sigma_r<\alpha+1$ *and that* (2.2) (a) *is replaced by*

$$a_n = O_R\left[l_n^{\alpha}(l_n-l_{n-1})\right],\ l_n-l_{n-1} = O\left(l_n^{(r-\alpha+\sigma_r)/(r+1)}\right). \tag{4.1}$$

Then

$$\sigma_k \leqslant \frac{(r-k)(\alpha+1) + (k+1)\sigma_r}{r+1} \quad (0 \leqslant k < r) . \tag{4.2}$$

(B) *In Theorem* I (B), *suppose that* $\sigma_r \geqslant \alpha+1$ *and that* (2.4) (a) *is replaced by*

$$a_n = O_R[l_n^\alpha (l_n - l_{n-1})] , \; l_n - l_{n-1} = o(l_n) . \tag{4.3}$$

Then

$$\sigma_k = \sigma_r \; (0 \leqslant k < r) . \tag{4.4}$$

Proof. The proof of part (A) is on the lines of that of Theorem III excepting that now there is no appeal to Theorem B. The proof of part (B) may need a further explanation as follows. The two conditions of (4.3) together imply $a_n = o_R(l^{\alpha+1}_n)$ which, along with the first condition of (4.3), readily gives us

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{a_n + a_{n+1} + \ldots + a_m}{l_n^{\alpha+1}} = o_R(1) , \; \varepsilon \to 0 .$$

The conclusion (4.4) now follows obviously from Theorem I (B) with alternative (2.4) (a) and $\rho = \alpha+1$.

The following deduction supplements the preceding and has been kindly suggested by Prof. Bosanquet.

DEDUCTION 2. *Suppose that, in Deduction* 1, *we replace* (4.1) *in* (A) *and* (4.3) *in* (B) *by the common hypothesis*

$$a_n = O_R[l_n^\alpha (l_n - l_{n-1})] , \; \sigma_r \geqslant 0 . \tag{4.5}$$

Then we have, for $0 \leqslant k < r$, EITHER (A) $\sigma_k \leqslant \alpha+1$, OR (B) $\sigma_k = \sigma_r$, *according as* $\sigma_r < \alpha+1$ *or* $\sigma_r \geqslant \alpha+1$.

Proof. (A) We choose γ such that $(0 \leqslant) \, \sigma_r < \gamma < \alpha+1$ and, as in (2.6), assume that $B^r(x) = o(x^r)$. Then we infer, from an application of Lemma 1,

$$A^r(x) = o(x^{r+\gamma}) = o(x^{r+\alpha+1+\delta}) \text{ for every } \delta > 0 . \tag{4.6}$$

On the other hand, our hypothesis on a_n gives us first $a_n = O_R(l_n^{\alpha+1}) = o_R(l_n^{\alpha+1+\delta})$ and then, as in the proof of part (B) of Deduction 1,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{a_n + a_{n+1} + \ldots + a_m}{l_n^{\alpha+1+\delta}} = o_R(1) , \; \varepsilon \to 0 . \tag{4.7}$$

From (4.6) and (4.7) we obtain, appealing first to Theorem A and then to Lemma 3,

$$A(x) = o(x^{\alpha+1+\delta}), \; A^k(x) = o(x^{k+\alpha+1+\delta}), \; 0 \leqslant k < r. \qquad (4.8)$$

Now Lemma 2 establishes the summability (R, l_n, k) of $\Sigma a_n l_n^{-(\alpha+1+\delta)}$, or, of $\Sigma a_n l_n^{-s}$ for $\sigma \geqslant \alpha+1+\delta$ with arbitrary $\delta > 0$. Hence $\sigma_k \leqslant \alpha+1$ as required.

(B) We now choose γ such that $(\alpha+1 \leqslant)\, \sigma_r < \gamma$ and note that $\alpha+1+\delta$ can be replaced by γ in (4.7) and (4.8), so that, arguing as before, we establish the summability (R, l_n, k), $0 \leqslant k < r$, of $\Sigma a_n l_n^{-\gamma}$ where $\gamma > \sigma_r$ is arbitrary. Hence $\sigma_k \leqslant \sigma_r$ while $\sigma_r \leqslant \sigma_k$ universally, i.e., $\sigma_k = \sigma_r$ as we wished to prove.

Deduction 3. *If, for the Dirichlet series* $\Sigma a_n l_n^{-s}$, $\sigma_r > -\infty$ *and* $\underline{\lim}\, l_n/l_{n-1} > 1$, *then* $\sigma_k = \sigma_r$ *for* $0 \leqslant k < r$.

Proof. The hypothesis $\underline{\lim}\, l_n/l_{n-1} > 1$ makes

$$a_{n+1} + a_{n+2} + \dots + a_m = 0 \quad \text{for} \quad l_n < l_m < l_n + \varepsilon l_n$$

if ε is sufficiently small and $n > n_0(\varepsilon)$. Hence, for any ρ, in particular, for $\rho \leqslant \sigma_r$,

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < l_n + \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^{\rho}} = o(1), \; \varepsilon \to 0.$$

The desired conclusion now follows from Theorem I (B) with alternative (2.4) (b).

In the above proof we have supposed that $\sigma_r < \infty$, the case $\sigma_r = \infty$ being trivial.

Concluding Remarks

A few remarks are offered in conclusion, supplementing some made in the beginning. Though Theorem A in one form is Karamata's (as already said), a particularization of it ([12], Corollary VI with Tauberian O-condition) is a much older theorem of Ananda-Rau's ([1], Theorem 16; [2], Theorem 4). Ananda Rau left open one case of his theorem which Bosanquet ([4], Theorems 2, 3), Minakshisundaram and Rajagopal ([10], Theorem 1 and Corollaries 1.1, 1.3; [11], Theorem A and Corollaries A_1, A_2) have independently settled, even for some extensions of Ananda Rau's theorem. The theorem mentioned at the outset as being due to Chandrasekharan and Minakshisundaram ([6], p. 21, Theorem 1.82) is, in fact, a further extension of one of the extensions of Ananda Rau's theorem given by

Bosanquet ([4], Theorem 3). In the present context, it is rather less effective than the completely independent two-fold result of Karamata's in the same direction ([9], Théorèmes 1a), 3f)), reformulated as Theorem A. That is to say, precisely, Theorem A gives rise to a basic converse theorem on abscissae of summability of general Dirichlet series (Theorem I of this paper) which is more natural and suggestive as well as more comprehensive than the like basic theorem resulting from the line of development followed by Chandrasekharan and Minakshisundaram ([6], p. 86, Theorem 3.71). [1])

I am indebted to Prof. Bosanquet for some very useful remarks on the original version of this paper which have led to the preparation of the present version.

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1) Indeed the Chandrasekharan-Minakshisundaram theorem just referred to is deducible from Theorem I, its case $\sigma_r < \alpha + \mu$ [or, case $\sigma_r \geqq \alpha + \mu$] from part (A) [or, part (B)] of Theorem I with hypothesis (2.2) (b) and $x^{\rho} = x^{\alpha} \{\theta(x)\}^{\mu}$, $\theta(x) = x^{(r-\alpha+\gamma)/(r+\mu)}$, $\sigma_r < \gamma < \alpha + \mu$ [or, hypothesis (2.4) (b) and $x^{\rho} = x^{\alpha+\mu}$].

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