§2. A BASIC THEOREM

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The theorem which follows supplies a basis for all the other theorems of this paper, whether by itself or not.

THEOREM I. (A) *For the Dirichlet series*

$$\sum_{1}^{\infty} \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau,$$

suppose that $\sigma_r < \rho$ *for some* $r > 0$. *Suppose also that there is a* γ *and an associated* $\theta(x)$ *such that*

$$\sigma_r < \gamma < \rho, \quad \theta(x) \equiv x^{1-(\rho-\gamma)/r}, \tag{2.1}$$

with

EITHER (a)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < \varepsilon\theta(l_n)} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^\rho} = o_R(1), \ \varepsilon \to 0,$$

OR (b)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < \varepsilon\theta(l_n)} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \ \varepsilon \to 0. \tag{2.2}$$

Then

$$\sigma_k \leqslant \frac{(r-k)\rho + k\sigma_r}{r} \quad (0 \leqslant k < r). \tag{2.3}$$

(B) *If* $\sigma_r \geqslant \rho$, *instead of* $\sigma_r < \rho$ *as in* (A), *and* ρ *is such that*

EITHER (a)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < \varepsilon l_n} \frac{a_n + a_{n+1} + \dots + a_m}{l_n^\rho} = o_R(1), \ \varepsilon \to 0,$$

OR (b)

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < \varepsilon l_n} \frac{|a_{n+1} + a_{n+2} + \dots + a_m|}{l_n^\rho} = o(1), \ \varepsilon \to 0, \tag{2.4}$$

then

$$\sigma_k = \sigma_r \ (0 \leqslant k < r). \tag{2.5}$$

Proof. (A) The proof is given below only for the case in which $\{a_n\}$ satisfies the hypothesis in alternative (2.2) (a), the remaining case of alternative (2.2) (b) being exactly similar.

By the definition of σ_r and that of γ in (2.1), $\Sigma a_n l_n^{-\gamma}$ is summable (R, l_n, r) to sum S (say), and so

$$b_1+b_2+\ldots = (a_1 l_1^{-\gamma}-S) + a_2 l_2^{-\gamma} + \ldots \text{ is summable (R, } l_n, r) \text{ to } 0,$$

i.e.,

$$B^r(x) = o(x^r)\,, \quad x \to \infty\,, \tag{2.6}$$

while it is easy to prove that

$$\overline{\lim_{n\to\infty}} \max_{l_n \leq l_m < l_n + \varepsilon\theta(l_n)} \frac{b_n + b_{n+1} + \ldots + b_m}{l_n^{\rho-\gamma}} = o_R(1)\,, \quad \varepsilon \to 0\,, \tag{2.7}$$

distinguishing between the case $\gamma \geqslant 0$ and $\gamma < 0$. If $\gamma \geqslant 0$, then

$$b_n + b_{n+1} + \ldots + b_m = a_n l_n^{-\gamma} + a_{n+1} l_{n+1}^{-\gamma} + \ldots + a_m l_m^{-\gamma} \; (n>1)$$

$$\leqslant l_n^{-\gamma} \max_{n \leq \nu \leq m} (a_n + a_{n+1} + \ldots + a_\nu)\,,$$

from which and (2.2) (a) we have (2.7) as an immediate consequence. On the other hand, if $\gamma < 0$, then

$$b_n + b_{n+1} + \ldots + b_m \leqslant l_m^{-\gamma} \max_{n \leq \nu \leq m} (a_\nu + a_{\nu+1} + \ldots + a_m)\,, \tag{2.8}$$

where

$$l_n \leqslant l_\nu \leqslant l_m < l_n + \varepsilon\,\theta(l_n) < (1+\varepsilon)\, l_n\,,$$

$$\frac{\theta(l_n)}{\theta(l_\nu)} = \left(\frac{l_n}{l_\nu}\right)^{(r-\rho+\gamma)/r} < (1+\varepsilon)^{|r-\rho+\gamma|/r} = K\,(\text{say})\,.$$

Hence (2.8) gives us

$$\max_{l_n \leq l_m < l_n + \varepsilon\theta(l_n)} \frac{b_n + b_{n+1} + \ldots + b_m}{l_n^{\rho-\gamma}}$$

$$\leqslant \frac{l_m^{-\gamma}}{l_n^{\rho-\gamma}} \max_{l_\nu \leq l_m < l_n + \varepsilon\theta(l_n)} (a_\nu + a_{\nu+1} + \ldots + a_m)$$

$$\leqslant \left(\frac{l_m}{l_\nu}\right)^{-\gamma} \left(\frac{l_\nu}{l_n}\right)^{\rho-\gamma} \max_{l_\nu \leq l_m < l_\nu + \varepsilon K\theta(l_\nu)} \frac{a_\nu + a_{\nu+1} + \ldots + a_m}{l_\nu^{\rho}}\,. \tag{2.9}$$

In (2.9),

$$\left(\frac{l_m}{l_\nu}\right)^{-\gamma} \leqslant 1 \,, \left(\frac{l_\nu}{l_n}\right)^{\rho-\gamma} < (1+\varepsilon)^{\rho-\gamma}$$

since $\gamma<0$, $\rho-\gamma>0$. Hence (2.9) in conjunction with hypothesis (2.2) (a) leads to (2.7) again.

After this we appeal to Theorem A with (1.1) and (1.2) (a) replaced by (2.6) and (2.7) respectively, to obtain

$$B(x) \equiv B(l_n) = o(l_n^{\rho-\gamma}) = o(x^{\rho-\gamma}) \,, \; l_{n+1} > x \geqslant l_n \to \infty \,.$$

From the last step and (2.6), we get, by using Lemma 3,

$$B^k(x) = o\left\{x^{\left(1-\frac{k}{r}\right)(\rho-\gamma)+\frac{k}{r}r}\right\} \text{ for } 0 \leqslant k < r$$

$$= o(x^{k+\beta}) \text{ where } \beta = \left(1-\frac{k}{r}\right)(\rho-\gamma) > 0 \,.$$

Hence $\Sigma a_n l_n^{-\gamma-\beta}$, being summable (R, l_n, r), is also summable (R, l_n, k) by Lemma 2, i.e., $\Sigma a_n l_n^{-s}$ is summable (R, l_n, k) for

$$\sigma \geqslant \gamma + \beta = \gamma + \left(1-\frac{k}{r}\right)(\rho-\gamma) = \left(1-\frac{k}{r}\right)\rho + \frac{k}{r}\gamma \,. \qquad (2.10)$$

Since $\gamma>\sigma_r$ may be taken as near to σ_r as we please [1]), (2.10) immediately gives us the conclusion (2.3).

In arriving at (2.3) we have tacitly assumed that $\sigma_r>-\infty$. When $\sigma_r=-\infty$, we still reach (2.3) in the sense that $\sigma_k=-\infty$ for $0\leqslant k<r$, as we may see by taking $\gamma=-G$ (G positive and arbitrarily large) in the preceding argument.

(B) As in (A), we confine ourselves to the hypothesis (2.4) (a), the treatment of (2.4) (b) being precisely similar. Defining as in (A)

$$b_1 + b_2 + \ldots \equiv (a_1 l_1^{-\gamma} - S) + a_2 l_2^{-\gamma} + \ldots \; (\gamma > \sigma_r) \,,$$

we see that (2.6) holds again, while (2.4) (a) implies

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} \frac{(b_n + b_{n+1} + \ldots + b_m)}{l_n^{\rho-\gamma}} = o_R(1) \,, \; \varepsilon \to 0 \,,$$

1) The truth of hypothesis (2.2) for some γ, $\sigma_r < \gamma < \rho$, implies its truth for any γ', $\sigma_r < \gamma' < \gamma$, so that γ may be replaced by γ'.

exactly as (2.2) (a) implies (2.7). Since now $\gamma > \sigma_r \geqslant \rho$, the above condition in its turn implies

$$\overline{\lim_{n\to\infty}} \max_{l_n \leqq l_m < l_n + \varepsilon l_n} (b_n + b_{n+1} + \ldots + b_m) = o_R(1), \quad \varepsilon \to 0 .$$

By Theorem A with hypothesis (1.2) (a) and $a = b = 0$, it follows that $\Sigma a_n l_n^{-s}$ is convergent for any σ such that $\sigma \geqslant \gamma > \sigma_r$ and therefore $\sigma_0 \leqslant \sigma_r$. But, in any case, $\sigma_0 \geqslant \sigma_k \geqslant \sigma_r$ for $0 \leqslant k < r$ and so we have the conclusion (2.5).

In the preceding argument we have supposed that $\sigma_r < \infty$ since $\sigma_r = \infty$ implies trivially $\sigma_k = \infty$.

§ 3. Applications to theorems of the Schnee-Landau type

Theorem II given next is the simplest of the theorems of the type mentioned above and it is a direct combination of Theorems I, B. Theorems V, VI are generalizations, respectively of Ananda-Rau's and Ganapathy Iyer's extensions of the Schnee-Landau theorem ([2], Theorem 9; [7], Theorem 10), as given by Chandrasekharan and Minakshisundaram ([6], pp. 88-9, Corollaries 3.73, 3.74). Theorems III, IV are apparently new counterparts of Theorems V, VI, the newness consisting in the replacement of the two-sided Tauberian conditions of the latter pair of theorems by analogous one-sided conditions suitably supplemented.

Theorem II. *Suppose that* (i) *the Dirichlet series,*

$$\sum_1^\infty \frac{a_n}{l_n^s}, \quad s = \sigma + i\tau ,$$

is summable (R, l_n, q) *for some* $q \geqslant 0$ *when* $\sigma > \rho$, (ii) *the sum-function* $f(s)$ *thus defined is regular for* $\sigma > \eta$ *when* $\eta < \rho$, *and satisfies the condition*

$$f(s) = O(|\tau|^r),\ r > 0, \quad \textit{uniformly for} \quad \sigma \geqslant \eta + \varepsilon > \eta,$$

(iii) *the coefficients* a_n *of the Dirichlet series satisfy* ONE *of the two alternatives* (a), (b) *of* (2.2), *but with* $\theta(x) \equiv x^{1-(\rho-\eta)/r}$. *Then the Dirichlet series is summable* (R, l_n, k), $0 \leqslant k < r$, *for*

$$\sigma > \frac{(r-k)\rho + k\eta}{r} .$$

Proof. By Theorem B, the Dirichlet series is summable (R, l_n, r'), $r' > r$, for $\sigma > \eta$ and hence $\sigma_{r'} \leqslant \eta < \rho$. Therefore it is evident from the proof of