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satisfying $1 \leqq p < 2 < q \leqq \infty$, the series (6.6) converges normally in $L^q_p(G)$ to T. Next, T is the limit in E of

$$S_r = \sum_{n=1}^{r} \omega_n T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every r, (ii) is easily derived. Finally, if $\hat{T}$ were a measure μ, it would necessarily be the case that supp $\mu \subseteq \bar{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(0) | = | \int_\Gamma \hat{u}_n \hat{v}_n \, d\mu | \\ &\leqq | \mu | (\bar{\Omega}), \end{aligned}$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if *either* $q \leqq 2$ *or* $p \geqq 2$, then every $T \in L^q_p(G)$ is such that $\hat{T}$ is a measure [and indeed a measure of the form $\psi \lambda_\Gamma$, where $\psi \in L^2_{loc}(\Gamma)$ if $q \leqq 2$ and $\psi \in L^p_{loc}(\Gamma)$ if $p \geqq 2$, and so $\psi \in L^2_{loc}(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

Part 3: Applications to Fourier series

§ 7. *Applications to divergence of Fourier series.*

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ, and λ_G the Haar measure on G, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, $\hat{f}$ will denote the Fourier transform of f; for any finite subset Δ of Γ,

$$S_\Delta f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \tag{7.1}$$

is the Δ-partial sum of the Fourier series of f; and sp (f) will stand for the spectrum of f, i.e., for the support $\operatorname{supp} \hat{f} = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$ of $\hat{f}$. The term "trigonometric polynomial" will frequently be abbreviated to "t.p.". In addition, Φ will denote the largest torsion subgroup of Γ ([7], (A.4)), and π the natural map of Γ onto Γ/Φ. If Δ denotes a subset of Γ, $[\Delta]$ will stand for the subgroup of Γ generated by Δ.

By a *(convergence) grouping* we shall mean a sequence $\mathscr{D} = (\Delta_j)_{j \in N} = (\Delta_j)$ of finite subsets Δ_j of Γ such that

$$\left.\begin{array}{l} \Delta_j \subseteq \Delta_{j+1} \quad (j \in N); \\[2ex] \bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0 \text{ is a subgroup of } \Gamma, \text{ said to be} \\ \qquad \textit{covered by } \mathscr{D}; \\[2ex] \text{for each } j \in N,\ \Delta_j = \Omega_j + \Lambda_j, \text{ where } \Lambda_j \text{ is a} \\ \qquad \text{nonvoid finite subset of } \Phi \text{ and } \Omega_j \text{ is a finite} \\ \qquad \text{subset of } \Gamma \text{ such that } \pi \mid \Omega_j \text{ is 1-1.} \end{array}\right\} \qquad (7.2)$$

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping $\mathscr{D}$ is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

7.2 EXAMPLES. (i) Let Γ_0 be any countable subgroup of Γ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in Z\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping $\mathscr{D}$ covering Γ_0 results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union equal to Γ_0. This grouping is of infinite type if and only if Γ_0 is infinite.

(ii) If G is connected, and if Γ_0 is any countable subgroup of Γ, then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) Γ_0 is an ordered group isomorphic to a discrete subgroup of R. Assuming $\Gamma_0 \neq \{0\}$, Γ_0 has a smallest positive element γ_0 and $\Gamma_0 = \{n\gamma_0 : n \in Z\}$. A natural grouping $\mathscr{D}$ covering Γ_0 is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{n\gamma_0 : n \in Z, |n| \leq j\}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions f on G satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in N}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j\to\infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions f. However, our concern rests with the possibility of constructing continuous functions f on G satisfying

$$\mathrm{sp}\,(f) \subseteq \Gamma_0, \ \overline{\lim_{j\to\infty}}\ \mathrm{Re}\, S_{\Delta_j} f(0) = \infty. \tag{7.3}$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether G is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space $E = C(G)$ of continuous complex valued functions on G [with norm $\|\cdot\|$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mapsto \mathrm{Re}\, S_\Delta f(0) = \mathrm{Re} \int_G D_\Delta f \, d\lambda_G, \tag{7.4}$$

where D_Δ stands for the "Dirichlet function"

$$D_\Delta = \sum_{\gamma \in \Delta} \bar{\gamma}, \tag{7.5}$$

shows that the problem hinges on the existence of groupings $\mathscr{D}$ for which

$$\rho_j = \| D_{\Delta_j} \|_1 = \int_G | D_{\Delta_j} | \, d\lambda_G \to \infty. \tag{7.6}$$

Accordingly, and in view of the fact ([7], (24.26)) that G is 0-dimensional if and only if Γ coincides with Φ, it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

(i) G is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions f on G satisfying (7.3). In particular [cf. Example 7.2 (i)], if Γ_0 is any countably infinite subgroup of Γ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j\in N}$ is any increasing sequence of finite subsets of Γ_0 with union Γ_0, we can construct a continuous f on G satisfying (7.3).

(ii) G is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup Γ_0 of Γ, there are groupings $\mathscr{D} = (\Delta_j)$ covering Γ_0, in which $\Omega_j = \{0\}$ and $\Delta_j = \Lambda_j$ is a finite subgroup of Γ_0, and for which

$$f = \lim_{j\to\infty} S_{\Delta_j} f$$

uniformly on G for every continuous f satisfying sp $(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 Remark. Perhaps it should be stressed here that, if Γ_0 is any infinite subgroup of Γ, there is no obstacle to constructing continuous functions f such that sp $(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of Γ_0 for which

$$\lim_j S_{\Delta_j} f(0) = \infty.$$

[One has in fact only to construct a continuous f such that sp $(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \infty$; it is then trivial that there exist finite subsets Δ of Γ_0 for which $|S_\Delta f(0)|$ is arbitrarily large, so that we can choose a sequence (Δ_j) for which $\Delta_j \subseteq \Delta_{j+1}$ and $|S_{\Delta_j} f(0)| \to \infty$ with j.] However, the sets Δ_j obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ covering Γ_0 and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j, \nu}$ in two indeterminates over the real field (ν being a suitable fixed integer not less than 36 and p_j any positive number not less than $\| D_{\Delta_j} \|_\infty$) such that, for suitable unimodular complex numbers ξ_j, the t.p.s

$$Q_j = \xi_j \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_j, \nu} (D_{\Delta_j}, \bar{D}_{\Delta_j})$$

satisfy

$$\left.\begin{aligned} &\| Q_j \| \leqq 1, \; sp\,(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0, \\ &S_{\Delta_j} Q_j(0) = \textstyle\int_G D_{\Delta_j} Q_j \, d\lambda_G \text{ is real and } \geqq \tfrac{1}{2} \rho_j. \end{aligned}\right\} \tag{7.7}$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n \in N}$ of positive integers so that

$$\left.\begin{aligned} &S_{\Delta_{j_n}} Q_{j_n}(0) \text{ is real and } > n^3, \\ &j_n < j_{n+1}, \; sp\,(Q_{j_n}) \subseteq \Gamma_0. \end{aligned}\right\} \tag{7.8}$$

Accordingly, the t.p.s

$$u_n = n^{-2}\, Q_{j_n}$$

satisfy the conditions

$$\left.\begin{array}{l} \mathrm{sp}\,(u_n) \subseteq \Gamma_0,\ \sum_{n=1}^{\infty} \| u_n \| < \infty \\ \\ S_{\Delta_{j_n}} u_n\,(0) \text{ is real and } > n. \end{array}\right\} \tag{7.9}$$

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < \ldots$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p\, u_{n_p}$$

is continuous and satisfies

$$sp\,(f) \subseteq \Gamma_0,\ \lim_{p \to \infty} \mathrm{Re}\, S_{\Delta_{j_{n_p}}} f(0) = \infty. \tag{7.10}$$

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing $\mathrm{Re}\, S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. *Discussion of case (i) :* G *not* 0-*dimensional*

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\varnothing \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant $k > 0$)

$$\| D_\Delta \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = | \, \Omega \, |$ (the cardinal number of Ω) is sufficiently large.

8.2 Proof of (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ / Φ. Likewise identify the dual of $K = G/H$ with Φ ([7], (24.11)).

We then have

$$\| D_\Delta \|_1 = \int_G \Big| \sum_{\gamma \in \Delta} \gamma \Big| \, d\lambda_G$$

$$= \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta \in \Omega} \sum_{\phi \in \Lambda} \theta(x+y)\, \phi(x+y) \Big| \, d\lambda_H(y),$$

the inner integral being viewed as a function of $\bar{x} = x+H$ Thus, writing $\bar{\theta}$ for $\pi(\theta)$ and noting that $\phi(y) = 1$ for $\phi \in \Lambda \subseteq \Phi$ and $y \in H$, we obtain

$$\| D_\Delta \|_1 = \int_{G/H} d\lambda_{G/H}(\bar{x}) \int_H \Big| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \Big| \, d\lambda_H(y), \tag{8.3}$$

where

$$\alpha(\theta, x) = \theta(x) \sum_{\phi \in \Lambda} \phi(x).$$

Now, since the dual of H (namely Γ/Φ) is torsion-free ([7], (A.4)), Theorem A of [8] shows that (for a suitable absolute constant $k > 0$) we have

$$\int_H \Big| \sum_{\theta \in \Omega} \alpha(\theta, x) \bar{\theta}(y) \Big| \, d\lambda_H(y) \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\theta \in \Omega} | \alpha(\theta, x) |$$

$$= k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \Big| \sum_{\phi \in \Lambda} \phi(\bar{x}) \Big|, \tag{8.4}$$

since $|\theta(x)| = 1$ and $\phi(x)$ depends only $\bar{x}$. By (8.3) and (8.4),

$$\| D_\Delta \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \int_{G/H} \Big| \sum_{\phi \in \Lambda} \phi(\bar{x}) \Big| \, d\lambda_{G/H}(\bar{x}). \tag{8.5}$$

Since $\Lambda \neq \varnothing$, the remaining integral is not less than the maximum modulus of the Fourier transform of the function $\bar{x} \mapsto \sum_{\phi \in \Lambda} \phi(\bar{x})$, i.e., is not less than unity. Thus, (8.2) follows from (8.5).

8.3 Proof of 7.4 (i). The conclusions stated in case (i) of 7.4 are now almost immediate. If $\mathscr{D} = (\Delta_j)_{j \in N}$ is a grouping of infinite type covering Γ_0, $|\pi(\Delta_j)| \to \infty$ and so, since $\Lambda_j \subseteq \Phi$, $|\pi(\Omega_j)| \to \infty$. Then (8.2) shows that (7.6) is satisfied, and it remains only to refer to 7.6.

8.4 Supplementary remarks. The fact that, when G is not 0-dimensional, (7.6) holds for suitable subgroups Γ_0 of Γ and suitable groupings $\mathscr{D} = (\Delta_j)_{j \in N}$ covering Γ_0 can be derived without appeal to Theorem A of [8]. To do this, it suffices to take $\gamma_k \in \Gamma \setminus \Phi$ ($k = 1, 2, \dots, m$) such that the family $(\gamma_k)_{1 \leqq k \leqq m}$ is independent (see [7], (A.10)), define

$$\Gamma_0 = \{\textstyle\sum_{k=1}^{m} n_k \gamma_k : n_k \in Z \text{ for } k = 1, 2, \dots, m\},$$

and make use of the formula

$$\int_G F(\gamma_1(x), \dots, \gamma_m(x)) d\gamma_G(x)$$
$$= (2\pi)^{-m} \int_0^{2\pi} \dots \int_0^{2\pi} F(e^{it}, \dots, e^{it_m}) dt_1 \dots dt_m, \qquad (8.6)$$

valid for every $F \in C(T^m)$, where T denotes the circle group. (Recall that $\sum_{k=1}^{m} n_k \gamma_k$ denotes the character $x \mapsto \gamma_1(x)^{n_1} \dots \gamma_m(x)^{n_m}$ of G.) It then appears that (7.6) holds when one takes

$$\Delta_j = \{\textstyle\sum_{k=1}^{m} n_k \gamma_k : |n_k| \leqq r_{j,k} \text{ for } k = 1, 2, \dots, m\},$$

where the $r_{j,k}$ are positive integers satisfying $r_{j,k} < r_{j,k+1}$ and $\lim_{j \to \infty} r_{j,k} = \infty$. Moreover, when $m = 1$, the Cohen-Davenport result (essentially Theorem A of [8] for the case $G = T$) shows that (7.6) holds for every grouping $\mathscr{D}$ covering Γ_0.

The verification of (8.6) is simple. First note that, if G and G' are compact groups, and if ϕ is a continuous homomorphism of G into G', then

$$\int_G (F \circ \phi) d\lambda_G = \int F d\lambda_{\phi(G)} \qquad (8.7)$$

for every $F \in C(G')$. (This is a consequence of the fact that $F \mapsto \int_G (F \circ \phi) d\lambda_G$ is invariant under translation by elements of $\phi(G)$, combined with the uniqueness of the normalised Haar measure on a compact group.) Taking $G' = T^m$ and $\phi : x \mapsto (\gamma_1(x), \dots, \gamma_m(x))$, the stated conditions on the γ_k are just adequate to ensure that the annihilator in Z^m (identified in the canonical fashion with the dual of T^m) of $\phi(G)$ is $\{(0, \dots, 0)\}$ and so ([7], (24.10)) that $\phi(G) = T^m$. Accordingly, (8.6) appears as a special case of (8.7).

It is perhaps worth indicating that special cases of (8.7) can be exploited in other ways. For example, suppose more generally that κ is an arbitrary nonvoid set and that $(\gamma_k)_{k \in \kappa}$ is a finite or infinite independent family of elements of $\Gamma \setminus \Phi$. Denote by Γ_0 the subgroup of Γ generated by $\{\gamma_k : k \in \kappa\}$. Taking $G' = T^\kappa$ and $\phi : x \mapsto (\gamma_k(x))_{k \in \kappa}$, one may use (8.7) in a similar fashion to show that there is an isometric isomorphism $F \leftrightarrow F \circ \phi = f$ between $L^p(T^\kappa)$ (or $C(T^\kappa)$) and the subspace of $L^p(G)$ (or $C(G)$) formed of those $f \in L^p(G)$ or $C(G)$) such that $\mathrm{sp}(f) \subseteq \Gamma_0$. Moreover, if one identifies in the canonical fashion the dual of T^κ with the weak direct product Z^{κ^*}, the said isomorphism is such that $\hat{F} = \hat{f} \circ \phi'$, where ϕ' is the isomorphism of Z^{κ^*} onto Γ_0 defined by $(n_k) \to \sum_{k \in \kappa} n_k \gamma_k$.

One consequence of this may be expressed roughly as follows: If the compact Abelian group G is such that $\Gamma \setminus \Phi$ contains an independent family of (finite or infinite) cardinality m, then Fourier series on G behave, in respect of convergence or summability, no better than do Fourier series on T^m.

Another consequence is that, if Δ is a subset of Γ_0, then Δ is a Sidon (or $\Lambda(p)$) subset of Γ if and only if $\phi'^{-1}(\Delta)$ is a Sidon (or $\Lambda(p)$) subset of Z^{κ^*}.

8.5 Further results. Theorem A of [8] implies something stronger than (8.2), namely: if ω is any complex-valued function on Γ such that

$$\omega(\gamma + \phi) = \omega(\gamma) \quad (\gamma \in \Gamma, \phi \in \Phi), \tag{8.8}$$

so that ω can be regarded as a function on Γ/Φ, and if we write

$$D_\Delta^\omega = \sum_{\gamma \in \Delta} \omega(\gamma)\,\bar{\gamma}, \; S_\Delta^\omega f = \sum_{\gamma \in \Delta} \omega(\gamma)\,\hat{f}(\gamma), \tag{8.9}$$

then, for $\Delta = \Omega + \Lambda$ as in (8.1), we have

$$\| D_\Delta^\omega \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}} \min_{\gamma \in \Omega} | \omega(\gamma) | \tag{8.10}$$

provided $N = | \Omega |$ is sufficiently large.

So, if we can arrange for $\Omega = \Omega_j$ to vary in such a way that the right-hand side of (8.10) tends to infinity with j, the substance of 7.6 will lead to a continuous f satisfying $\mathrm{sp}(f) \subseteq \Gamma_0$ and

$$\overline{\lim_{j \to \infty}} \operatorname{Re} S_{\Delta_j}^\omega f(0) = \infty. \tag{8.11}$$

Taking the most familiar case, in which $G = T$, $\Gamma = Z$ and $\Phi = \{0\}$, and supposing $\Delta = \Omega$ to range over a sequence (Δ_j) of finite subsets of Z such that, if $N_j = | \Delta_j |$,

$$\lim_j \left(\frac{\log N_j}{\log \log N_j} \right)^{\frac{1}{4}} \min_{n \in \Delta_j} | \omega(n) | = \infty,$$

the construction will lead to a continuous f on T such that

$$\overline{\lim_j} \operatorname{Re} S_{\Delta_j}^\omega f(0) = \infty.$$

In particular, taking $\Delta_j = \{n \in Z : 2^j \leqq n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n \in Z} \frac{\pm \hat{f}(n)}{(\log(2+|n|))^{\alpha}}$$

diverges for any preassigned distribution of signs $\pm$ and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

§ 9. *Discussion of case (*ii*) :* G *0-dimensional*

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in G formed of compact open subgroups W. For each such W the annihilator $\Delta = W^{\circ}$ in Γ of W is a finite subgroup of Γ. Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \tag{9.1}$$

Then k_W is continuous, $k_W \geqq 0$, $\int_G k_W \, d\lambda_G = 1$. The transform $\hat{k}_W$ of k_W is plainly equal to unity on Δ. On the other hand, since W is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} \, d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} \, d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} \, d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that $\hat{k}_W(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus $\hat{k}_W$ is the characteristic function of Δ, and so

$$k_W = D_{W^{\circ}}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \leqq p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_W S_{W^{\circ}} f \tag{9.3}$$

in $L^p(G)$; and that (9.3) holds uniformly for any continuous f.

9.2 Proof of 7.4 (ii). If Γ_0 is any countably infinite subgroup of Γ we can choose a sequence W_j of compact open subgroups of G such that

$W_{j+1} \subseteq W_j$ and $\Gamma_0 \subseteq \bigcup_{j=1}^{\infty} W_j^\circ$, where W_j° is a finite subgroup of Γ and $W_j^\circ \subseteq W_{j+1}^\circ$. The $\Delta_j = W_j^\circ \cap \Gamma_0$ satisfy (7.2) and, from (9.3),

$$f = \lim_j S_{\Delta_j} f \tag{9.4}$$

uniformly for any continuous f with $\operatorname{sp}(f) \subseteq \Gamma_0$. This verifies the statements made in 7.4 (ii).

9.3 By using the results in [3], more can be said in case (ii) of 7.4; cf. [3], Theorem (2.9) and Example (4.8).

Let $f \in L^1(G)$ and let Γ_0 be any countable subgroup of Γ containing $\operatorname{sp}(f)$. Choose the W_j as in 9.2. Then, apart from the fact that (W_j) is not in general a base at 0 in G (they can be chosen to be so if and only if G is first countable), (W_j) is an open-compact D''-sequence ([3], p. 188). The proof of Theorem (2.5) of [3] is easily modified to show that

$$f(x) = \lim_{j \to \infty} S_{W_j^\circ} f(x) \tag{9.5}$$

holds for almost all $x \in G$. Moreover, Theorem (2.7) of [3] applies to show that the majorant function

$$S^* f(x) = \sup_{j \in N} \left| S_{W_j^\circ} f(x) \right| \tag{9.6}$$

satisfies the estimates

$$\| S^* f \|_p \leqq 2 (p (p-1)^{-1})^{\frac{1}{p}} \| f \|_p \quad (1 < p < \infty) \tag{9.7}$$

$$\| S^* f \|_1 \leqq 2 + 2 \textstyle\int_G |f| \log^+ |f| \, d\lambda_G, \tag{9.8}$$

$$\| S^* f \|_p \leqq 2 (1-p)^{\frac{1}{p}} \| f \|_1 \quad (0 < p < 1). \tag{9.9}$$

In particular, the convergence in (9.5) is dominated whenever

$$|f| \log^+ |f| \in L^1(G).$$

A more immediate consequence of (9.1) and (9.2) is a strong version of localisability of the convergence of Fourier series: if $f \in L^1(G)$ vanishes a.e. on some neighbourhood of $x_0 \in G$, we can choose the W_j so that $S_{\Delta_j} f(x_0) = 0$ for every sufficiently large j. [A suitable choice of W_j may be made once for all, independent of f, if G is first countable.] Nothing similar is true for general G; see, for example, [11], Vol. II, pp. 304-305.

§ 10. *Concerning the polynomials* Q_j.

There is no difficulty in making fairly explicit the construction of t.p.s Q_j of the type employed in 7.6.

For $p > 0$, $t \geqq 0$ define

$$h_p(t) = \begin{cases} 1 & \text{if } t \leqq p, \\ 2\left(1 - \dfrac{t}{2p}\right) & \text{if } p \leqq t \leqq 2p, \\ 0 & \text{if } t \geqq 2p. \end{cases} \tag{10.1}$$

For all complex z define

$$f_p(z) = \begin{cases} 0 & \text{if } z = 0, \\ |z|^{-1} \bar{z} h_p(|z|) & \text{if } z \neq 0. \end{cases} \tag{10.2}$$

Write

$$\left.\begin{aligned} E_n(z) &= \pi^{-1} n \exp(-n|z|^2, \\ P_{n,k}(z) &= \pi^{-1} n \sum_{j=0}^{k} \frac{(-1)^j}{j!} (n|z|^2)^j \end{aligned}\right\} \tag{10.3}$$

Let μ denote Lebesgue measure on C (identified with R^2 in the canonical fashion).

It is then routine to verify that

$$\left.\begin{aligned} &\| E_n * f_p \|_\infty \leqq \| f_p \|_\infty = 1, \\ &\lim_{n \to \infty} E_n * f_p = f_p \end{aligned}\right\} \tag{10.4}$$

uniformly on any compact set omitting 0. From this it follows that to every $p > 0$ and every positive integer v correspond positive integers $\bar{n}(p, v)$, $\bar{k}(p, v)$ such that

$$\left.\begin{aligned} &\left| |z|^{-1} \bar{z} - f_p * P_{\bar{n}, \bar{k}}(z) \right| \leqq \frac{1}{v} \text{ for } \frac{1}{v} \leqq |z| \leqq p, \\ &|f_p * P_{\bar{n}, \bar{k}}(z)| \leqq 1 + \frac{1}{v} \text{ for } |z| \leqq p. \end{aligned}\right\} \tag{10.5}$$

Now

$$f_p * P_{\bar{n}, \bar{k}}(z) = q_{p,v}(z, \bar{z}), \tag{10.6}$$

where

$$q_{p,\nu}(X, Y) = \pi^{-1}\,\bar{n}(p, \nu) \sum_{j=0}^{\bar{k}(p,\nu)} \frac{(-\bar{n}(p, \nu))^j}{j!} \sum_{l=0}^{j} \sum_{m=0}^{j} \binom{j}{l}\binom{j}{m} X^l\, Y^m$$

$$(-1)^{l+m} \int \zeta^{j-l}\bar{\zeta}^{j-m} f_p(\zeta)\, d\mu(\zeta)$$

$$= \sum_{l,m=0}^{\bar{k}(p,\nu)} C_{p,\nu}(l, m)\, X^l\, Y^m. \tag{10.7}$$

It is easily verifiable that the $C_{p,\nu}(l, m)$ are real-valued.

If θ is a bounded measurable function on G and

$$\overset{\circ}{Q}_{p,\nu} = q_{p,\nu}(\theta, \bar{\theta}),\ p \geqq \|\,\theta\,\|_\infty, \tag{10.8}$$

we have from (10.5)

$$\left.\begin{aligned} &\left|\,|\,\theta\,|^{-1}\,\bar{\theta} - \overset{\circ}{Q}_{p,\nu}\right| < \frac{1}{\nu} \text{ whenever } |\,\theta\,| \geqq \frac{1}{\nu}, \\ &|\,\overset{\circ}{Q}_{p,\nu}\,| \leqq 1 + \frac{1}{\nu} \text{ everywhere on } G. \end{aligned}\right\} \tag{10.9}$$

If θ is a t.p., then $\overset{\circ}{Q}_{p,\nu}$ is a t.p. and

$$\operatorname{sp}(\overset{\circ}{Q}_{p,\nu}) \subseteq [\operatorname{sp}(\theta)]. \tag{10.10}$$

From (10.9) we obtain

$$\left|\,|\,\theta\,| - \theta\,\overset{\circ}{Q}_{p,\nu}\right| \leqq \begin{cases} \nu^{-1}\,|\,\theta\,| \text{ whenever } |\,\theta\,| > \dfrac{1}{\nu}, \\ \left(2+\dfrac{1}{\nu}\right)|\,\theta\,| \text{ everywhere,} \end{cases}$$

whence it follows that, if $\theta \neq 0$,

$$\begin{aligned} \left|\int_G \theta\,\overset{\circ}{Q}_{p,\nu}\, d\lambda G\right| &\geqq (1-\nu^{-1})\,\|\,\theta\,\|_1 - \nu^{-1}(2+\nu^{-1}) \\ &\geqq (1-2\nu^{-\frac{1}{2}})\,\|\,\theta\,\|_1 \end{aligned} \tag{10.11}$$

provided $\nu \geqq 9\,\|\,\theta\,\|_1^{-2}$.

Taking $\theta = D_{\Delta_j}$ and $p_j \geqq \|\,D_{\Delta_j}\,\|$, the trigonometric polynomials

$$Q_j' = \left(1+\frac{1}{\nu}\right)^{-1} \overset{\circ}{Q}_{p_j,\nu} = \left(1+\frac{1}{\nu}\right)^{-1} q_{p_j,\nu}(D_{\Delta_j}, \bar{D}_{\Delta_j}) \tag{10.12}$$

are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left.\begin{aligned} \| Q_j' \| &\leqq 1, \\ \operatorname{sp}(Q_j') &\subseteq [\Delta_j], \\ \left| \int v\, D_{\Delta_j}\, Q_j'\, d\lambda_G \right| &\geqq (1 - 3v^{-\frac{1}{2}}) \| D_{\Delta_j} \|_1 \end{aligned}\right\} \tag{10.13}$$

provided v is chosen $\geqq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geqq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers ξ_j such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

Appendix

Rudin-Shapiro sequences

A.1 Notations and definitions. As hitherto, all topological groups G are assumed to be Hausdorff; and, for any locally compact group G, λ_G will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on G having compact supports.

If X and Y are topological groups, Hom (X, Y) denotes the set of continuous homomorphisms of X into Y.

Suppose henceforth G to be locally compact. As in 5.1, if $k \in C_c(G)$, T_k will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the (p, q)-norm of this operator, i.e., the smallest real number $m \geqq 0$ such that

$$\| f * k \|_q \leqq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if G is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where Γ is the character group of G and $\hat{k}$ is the Fourier transform of k. (Something similar is true whenever G is compact, but we shall not use this.)

U-RS-sequences on G are as defined in 5.4.