2. The main theorem

2. THE MAIN THEOREM

Let us first give some definitions. In the following, all functions are complex valued and continuous. Let ω denote Lebesgue measure on Ω_d, normalized so that $\omega(\Omega_d) = 1$.

For functions f, g and measures φ on Ω_d we write $(f, g) := \int f\bar{g}\,d\omega$ and $(f, \varphi) := \int f d\varphi$, where the integrals are extended over Ω_d. For $n = 0, 1, 2, \ldots$ let $\mathfrak{H}_n$ denote the complex vector space of spherical harmonics of degree n on Ω_d; let $N_{d,n}$ be its dimension. If f is a function on Ω_d, we say that $\mathfrak{H}_n$ *occurs in* f if and only if the orthogonal projection of f onto $\mathfrak{H}_n$ does not vanish, i.e. if $(f, Y_n) \neq 0$ for some spherical harmonic Y_n of degree n (or, equivalently, if $\int f(u)\, C_n^\nu(\langle u, v\rangle)\, d\omega(u)$ does not vanish identically, where C_n^ν is the Gegenbauer polynomial of degree n and order $\nu = \frac{1}{2}(d-2)$). Analogously, we say that $\mathfrak{H}_n$ occurs in the measure φ if and only if $(Y_n, \varphi) \neq 0$ for some $Y_n \in \mathfrak{H}_n$. If f is a function on Ω_d and $\delta \in SO(d)$ is a rotation, the *left translate* δf of f by δ is defined by $(\delta f)(u) = f(\delta^{-1} u)$ for $u \in \Omega_d$.

THEOREM 2.1. *Let* f *be a continuous function and* φ *a measure on* Ω_d. *In order that* $(\delta f, \varphi) = 0$ *for each* $\delta \in SO(d)$, *it is necessary and sufficient that none of the spaces* $\mathfrak{H}_n$, $n \in \{0, 1, 2, \ldots\}$, *occurs in both*, f *and* φ.

We remark that this theorem, together with its proof to be given below, carries over to the following more general situation: $SO(d)$ and Ω_d may be replaced, respectively, by a compact connected topological group G and by the homogeneous manifold G/K, where $K (= SO(d-1)$ in our case) is a closed subgroup of G. The rôle of the spherical harmonics is then played by their natural generalizations. We do not write down this generalization explicitly since we do not know any application of it.

Proof of Theorem 2.1. Let $\{ Y_{ni}; i=1, \ldots, N_{d,n} \}$ be an orthonormal basis of $\mathfrak{H}_n (n = 0, 1, 2, \ldots)$. Let us first suppose that f is a finite sum of spherical harmonics,

$$f = \sum_{n=0}^{k} \sum_{j=1}^{N_{n,d}} (f, Y_{nj})\, Y_{nj}\,. \tag{2.1}$$

Since $\mathfrak{H}_n$ is invariant under the action of $SO(d)$ by left translation, we have a relation

$$Y_{nj}(\delta^{-1} u) = \sum_{i=1}^{N_{d,n}} t^n_{ij}(\delta)\, Y_{ni}(u) \tag{2.2}$$

for each $\delta \in SO(d)$, by which continuous functions t^n_{ij} on $SO(d)$ are defined. It is well known that, for each $n \in \{0, 1, 2, \ldots\}$, the mapping $\delta \to (t^n_{ij}(\delta))$ is a unitary, irreducible matrix valued representation of the group $SO(d)$. From (2.1) and (2.2) we get

$$(\delta f, \varphi) = \sum_{n=0}^{k} \sum_{i,j=0}^{N_{d,n}} t^n_{ij}(\delta)(f, Y_{nj})(Y_{ni}, \varphi) . \tag{2.3}$$

If f is an arbitrary continuous function on Ω_d, then f can be uniformly approximated by a sequence $f_1, f_2, \ldots$, where each f_k is a finite sum of spherical harmonics of those degrees n only, for which $\mathfrak{H}_n$ occurs in f (see, e.g., Weyl [22], p. 499).

Let us now suppose that $\mathfrak{H}_n$ does not occur in both, f and φ $(n = 0, 1, 2, \ldots)$. Approximate f as explained above. Then if $(Y_{ni}, \varphi) \neq 0$ for some n and some $i \in \{1, \ldots, N_{d,n}\}$, the space $\mathfrak{H}_n$ does not occur in f. Therefore we have $(f_k, Y_{nj}) = 0$ $(k = 1, 2, \ldots)$ for each $j \in \{1, \ldots, N_{d,n}\}$, since f_k is a finite sum of spherical harmonics of degrees other than n. This shows that $(f_k, Y_{nj})(Y_{ni}, \varphi) = 0$ for each possible choice of k, n, i, j, and hence $(\delta f_k, \varphi) = 0$ by (2.3). For $k \to \infty$ we get $(\delta f, \varphi) = 0$, which proves one half of the theorem.

In order to prove the other direction of Theorem 2.1, we multiply equation (2.3) by $\overline{t^n_{km}(\delta)}$ and integrate over $SO(d)$ with respect to the normalized Haar measure μ. Using one of the well known orthogonality relations for the matrix elements of unitary, irreducible representations of a compact group, namely

$$N_{d,n} \int_{SO(d)} t^n_{ij}(\delta)\, \overline{t^n_{km}(\delta)}\, d\mu(\delta) = \delta_{ik}\, \delta_{jm} ,$$

we arrive at

$$N_{d,n} \int_{SO(d)} (\delta f, \varphi)\, \overline{t^n_{km}(\delta)}\, d\mu(\delta) = (f, Y_{nm})(Y_{nk}, \varphi) ,$$

provided f is a finite sum of spherical harmonics. By approximation, this holds for arbitrary continuous f. If now (1.6) is assumed, we get $(f, Y_{nm})(Y_{nk}, \varphi) = 0$ for $n = 0, 1, 2, \ldots$ and $k, m \in \{1, \ldots, N_{d,n}\}$, which shows that $\mathfrak{H}_n$ does not occur in both, f and φ. Theorem 2.1 is proved.