

## 2. The main theorem

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## 2. THE MAIN THEOREM

Let us first give some definitions. In the following, all functions are complex valued and continuous. Let  $\omega$  denote Lebesgue measure on  $\Omega_d$ , normalized so that  $\omega(\Omega_d) = 1$ .

For functions  $f, g$  and measures  $\varphi$  on  $\Omega_d$  we write  $(f, g) := \int f \bar{g} d\omega$  and  $(f, \varphi) := \int f d\varphi$ , where the integrals are extended over  $\Omega_d$ . For  $n = 0, 1, 2, \dots$  let  $\mathfrak{H}_n$  denote the complex vector space of spherical harmonics of degree  $n$  on  $\Omega_d$ ; let  $N_{d,n}$  be its dimension. If  $f$  is a function on  $\Omega_d$ , we say that  $\mathfrak{H}_n$  occurs in  $f$  if and only if the orthogonal projection of  $f$  onto  $\mathfrak{H}_n$  does not vanish, i.e. if  $(f, Y_n) \neq 0$  for some spherical harmonic  $Y_n$  of degree  $n$  (or, equivalently, if  $\int f(u) C_n^v(\langle u, v \rangle) d\omega(u)$  does not vanish identically, where  $C_n^v$  is the Gegenbauer polynomial of degree  $n$  and order  $v = \frac{1}{2}(d-2)$ ). Analogously, we say that  $\mathfrak{H}_n$  occurs in the measure  $\varphi$  if and only if  $(Y_n, \varphi) \neq 0$  for some  $Y_n \in \mathfrak{H}_n$ . If  $f$  is a function on  $\Omega_d$  and  $\delta \in SO(d)$  is a rotation, the left translate  $\delta f$  of  $f$  by  $\delta$  is defined by  $(\delta f)(u) = f(\delta^{-1}u)$  for  $u \in \Omega_d$ .

**THEOREM 2.1.** *Let  $f$  be a continuous function and  $\varphi$  a measure on  $\Omega_d$ . In order that  $(\delta f, \varphi) = 0$  for each  $\delta \in SO(d)$ , it is necessary and sufficient that none of the spaces  $\mathfrak{H}_n$ ,  $n \in \{0, 1, 2, \dots\}$ , occurs in both,  $f$  and  $\varphi$ .*

We remark that this theorem, together with its proof to be given below, carries over to the following more general situation:  $SO(d)$  and  $\Omega_d$  may be replaced, respectively, by a compact connected topological group  $G$  and by the homogeneous manifold  $G/K$ , where  $K (= SO(d-1)$  in our case) is a closed subgroup of  $G$ . The rôle of the spherical harmonics is then played by their natural generalizations. We do not write down this generalization explicitly since we do not know any application of it.

*Proof of Theorem 2.1.* Let  $\{Y_{ni}; i=1, \dots, N_{d,n}\}$  be an orthonormal basis of  $\mathfrak{H}_n$  ( $n = 0, 1, 2, \dots$ ). Let us first suppose that  $f$  is a finite sum of spherical harmonics,

$$(2.1) \quad f = \sum_{n=0}^k \sum_{j=1}^{N_{n,d}} (f, Y_{nj}) Y_{nj}.$$

Since  $\mathfrak{H}_n$  is invariant under the action of  $SO(d)$  by left translation, we have a relation

$$(2.2) \quad Y_{nj}(\delta^{-1} u) = \sum_{i=1}^{N_{d,n}} t_{ij}^n(\delta) Y_{ni}(u)$$

for each  $\delta \in SO(d)$ , by which continuous functions  $t_{ij}^n$  on  $SO(d)$  are defined. It is well known that, for each  $n \in \{0, 1, 2, \dots\}$ , the mapping  $\delta \rightarrow (t_{ij}^n(\delta))$  is a unitary, irreducible matrix valued representation of the group  $SO(d)$ . From (2.1) and (2.2) we get

$$(2.3) \quad (\delta f, \varphi) = \sum_{n=0}^k \sum_{i,j=0}^{N_{d,n}} t_{ij}^n(\delta) (f, Y_{nj})(Y_{ni}, \varphi).$$

If  $f$  is an arbitrary continuous function on  $\Omega_d$ , then  $f$  can be uniformly approximated by a sequence  $f_1, f_2, \dots$ , where each  $f_k$  is a finite sum of spherical harmonics of those degrees  $n$  only, for which  $\mathfrak{S}_n$  occurs in  $f$  (see, e.g., Weyl [22], p. 499).

Let us now suppose that  $\mathfrak{S}_n$  does not occur in both,  $f$  and  $\varphi$  ( $n = 0, 1, 2, \dots$ ). Approximate  $f$  as explained above. Then if  $(Y_{ni}, \varphi) \neq 0$  for some  $n$  and some  $i \in \{1, \dots, N_{d,n}\}$ , the space  $\mathfrak{S}_n$  does not occur in  $f$ . Therefore we have  $(f_k, Y_{nj}) = 0$  ( $k = 1, 2, \dots$ ) for each  $j \in \{1, \dots, N_{d,n}\}$ , since  $f_k$  is a finite sum of spherical harmonics of degrees other than  $n$ . This shows that  $(f_k, Y_{nj})(Y_{ni}, \varphi) = 0$  for each possible choice of  $k, n, i, j$ , and hence  $(\delta f_k, \varphi) = 0$  by (2.3). For  $k \rightarrow \infty$  we get  $(\delta f, \varphi) = 0$ , which proves one half of the theorem.

In order to prove the other direction of Theorem 2.1, we multiply equation (2.3) by  $\overline{t_{km}^n(\delta)}$  and integrate over  $SO(d)$  with respect to the normalized Haar measure  $\mu$ . Using one of the well known orthogonality relations for the matrix elements of unitary, irreducible representations of a compact group, namely

$$N_{d,n} \int_{SO(d)} t_{ij}^n(\delta) \overline{t_{km}^n(\delta)} d\mu(\delta) = \delta_{ik} \delta_{jm},$$

we arrive at

$$N_{d,n} \int_{SO(d)} (\delta f, \varphi) \overline{t_{km}^n(\delta)} d\mu(\delta) = (f, Y_{nm})(Y_{nk}, \varphi),$$

provided  $f$  is a finite sum of spherical harmonics. By approximation, this holds for arbitrary continuous  $f$ . If now (1.6) is assumed, we get  $(f, Y_{nm})(Y_{nk}, \varphi) = 0$  for  $n = 0, 1, 2, \dots$  and  $k, m \in \{1, \dots, N_{d,n}\}$ , which shows that  $\mathfrak{S}_n$  does not occur in both,  $f$  and  $\varphi$ . Theorem 2.1 is proved.